**Math 1151 - Test 2 Solutions**

**1.** Compute the derivatives of the following functions.

**a)** $f(x) = e^{\sin(x^2)}$

**Solution:** Use the chain rule: $f'(x) = e^{\sin(x^2)} \cdot \cos(x^2) \cdot 2x = 2xe^{\sin(x^2)} \cos(x^2)$.

**b)** $r(t) = \tan^{-1}\big(\ln(3t)\big)$

**Solution:** Use the chain rule: $r'(t) = \frac{1}{1+(\ln(3t))^2} \cdot \frac{1}{3t} \cdot 3 = \frac{1}{t(1+(\ln(3t))^2)}$.

**c)** $g(x) = e^x(2x+3)^x$

**Solution:** Logarithmic differentiation is necessary here, since the variable, $x$, appears in both the base and power of the second factor. Let $y = g(x) = e^x(2x+3)^x$, so that

$$\ln y = \ln(e^x(2x+3)^x) = \ln(e^x) + \ln((2x+3)^x) = x\ln e + x\ln(2x+3) = x + x\ln(2x+3).$$

Now, implicitly differentiate to obtain

$$\frac{1}{y}\frac{dy}{dx} = 1 + x \cdot \frac{2}{2x+3} + \ln(2x+3) \;\Rightarrow\; \frac{dy}{dx} = y\left(1 + \frac{2x}{2x+3} + \ln(2x+3)\right)$$

$$\Rightarrow\; g'(x) = \frac{dy}{dx} = e^x(2x+3)^x\left(1 + \frac{2x}{2x+3} + \ln(2x+3)\right).$$

Note: the derivative rule $\frac{d}{dx}(a^x) = (\ln a)a^x$ only applies when the base, $a$, is a constant, so this rule does not apply here.

**2.** Compute the following limits.

**a)** $\displaystyle\lim_{x\to 0^+} \sqrt{x}\ln x$

**Solution: ($=_*$ indicates an application of L'Hospital's Rule)**

$$\lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}} = 0 \cdot (-\infty) =_* \lim_{x\to 0^+} \frac{x^{-1}}{(-1/2)x^{-3/2}} = -2\lim_{x\to 0^+} x^{1/2} = 0.$$

**b)** $\displaystyle\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{2x}$

**Solution:** Let $y = (1+\frac{1}{x})^{2x}$, so that $\ln y = \ln\big((1+\frac{1}{x})^{2x}\big) = 2x\ln(1+\frac{1}{x})$. Computing the limit of $\ln y$, we get

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} 2x\ln\left(1+\frac{1}{x}\right) = \infty \cdot 0 = 2\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{x^{-1}}$$

$$=_* 2\lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}} \cdot \frac{-1}{x^2}}{\frac{-1}{x^2}} = 2\lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 2 \cdot 1 = 2.$$

So, the desired limit is then found as follows.

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{2x} = \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{\lim_{x\to\infty} \ln y} = e^2.$$

**3I.** Use linear approximations to estimate $\sqrt{15.9}$ by working through the following steps.

*Ia)* This number can best be estimated by using a linear approximation of the function $f(x) = \sqrt{x}$ at the point $a = \underline{16}$.

*Ib)* Given your answers to *Ia*, compute the best linear approximation - that is, the function $L(x)$ - to $f$ at the point $a$.

**Solution:** $L(x) = f'(a)(x-a) + f(a)$; $f'(x) = \frac{1}{2\sqrt{x}}$; $f(a) = 4$; $f'(a) = \frac{1}{8}$.

$$L(x) = \frac{1}{8}(x-16) + 4$$

*Ic)* Estimate the number $\sqrt{15.9}$ using *Ib*.

**Solution:**

$$\sqrt{15.9} = f(15.9) \approx L(15.9) = \frac{1}{8}(-0.1) + 4 = 4 - \frac{1}{80} = \frac{319}{80} = 3.9875.$$

**3II.** Use differentials to approximate $\sqrt{15.9}$ by working through the following steps.

*IIa)* This number can best be estimated by using the differential of the function $y = f(x) = \sqrt{x}$ at the point $x = 16$ and with a change in $x$ of $dx = \Delta x = -0.1$.

*IIb)* Using part *IIa*, compute $dy$, the corresponding approximate change in $y$.

**Solution:** $dy = f'(x)dx$, $f'(x) = \frac{1}{2\sqrt{x}}$, $f'(16) = \frac{1}{8} \Rightarrow dy = \frac{1}{8}(-0.1) = -\frac{1}{80}$.

*IIc)* Estimate the number $\sqrt{15.9}$ using *IIb*.

**Solution:** $dy \approx f(x+\Delta x) - f(x) \Rightarrow -\frac{1}{80} \approx f(16-0.1) - f(16) = \sqrt{15.9} - 4$
$\Rightarrow \sqrt{15.9} \approx 4 - \frac{1}{80} = \frac{319}{20} = 3.9875.$

**4.** Consider the curve given by the equation $x^4 - y\ln y = 2xy$. Find the equation of the normal line to this curve at the point $(0,1)$.

**Solution:** Implicitly differentiating, we obtain

$$4x^3 - y\frac{1}{y}\frac{dy}{dx} - (\ln y)\frac{dy}{dx} = 2x\frac{dy}{dx} + 2y$$

$$\Rightarrow \quad \frac{dy}{dx}(2x + 1 + \ln y) = 4x^3 - 2y \Rightarrow \frac{dy}{dx} = \frac{4x^3 - 2y}{2x + 1 + \ln y}.$$

Thus, the slope of the tangent line at $(0,1)$ is $-2$, which means that the slope of the normal line at that point is $\frac{1}{2}$. The equation of the normal line at $(0,1)$, then, is $y - 1 = \frac{1}{2}(x-0) \Rightarrow y = \frac{1}{2}x + 1$.

**5.** Consider the function $f(x) = x^4 e^{-x}$. (*NOTE: this problem extends to the following page.*)

**a)** Compute the limits $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$.
**Solution:** First, $\lim_{x\to-\infty} x^4 e^{-x} = \infty \cdot \infty = \infty$ (i.e. both $x^4$ and $e^{-x}$ go to $\infty$ as $x \to -\infty$). For the other direction, we have

$$\lim_{x\to\infty} x^4 e^{-x} = \infty \cdot 0 = \lim_{x\to\infty} \frac{x^4}{e^x} =_* \lim_{x\to\infty} \frac{4x^3}{e^x} =_* \lim_{x\to\infty} \frac{12x^2}{e^x} =_* \lim_{x\to\infty} \frac{24x}{e^x} =_* \lim_{x\to\infty} \frac{24}{e^x} = 0.$$

**b)** Determine - if any - the horizontal asymptotes for this function.
**Solution:** From part b, we conclude that $y = 0$ is a horizontal asymptote of $f$. (Recall: horizontal asymptotes correspond to finite limits at $\pm\infty$.)

**c)** Compute the first and second derivatives of $f$.
**Solution:**

$$f'(x) = x^4(-e^{-x}) + 4x^3 e^{-x} = -x^3 e^{-x}(x-4)$$

$$\begin{aligned} f''(x) &= (-x^3)(e^{-x})(1) + (-x^3)(-e^{-x})(x-4) + (-3x^2)(e^{-x})(x-4) \\ &= x^2 e^{-x}\Big(-x + x(x-4) - 3(x-4)\Big) \\ &= x^2 e^{-x}\Big(x^2 - 8x + 12\Big) = x^2 e^{-x}(x-2)(x-6) \end{aligned}$$

**d)** Find the critical points of $f$.
**Solution:** From part c, we see that $f'(x)$ is defined for all $x$ but equals 0 at $x = 0$, 4. These are the two critical points.

**e)** Determine the intervals over which $f$ is increasing and decreasing.
**Solution:** Testing sign of $f'$ over the intervals determined by the critical points, we obtain

$$x < 0 \implies f'(-1) = -5e < 0; \qquad 0 < x < 4 \implies f'(1) = 3e^{-1} > 0$$

$$x > 4 \implies f'(5) = -125e^{-5} < 0$$

Thus, $f$ is increasing on $(0, 4)$, and $f$ is decreasing on $(-\infty, 0)$ and $(4, \infty)$.

**f)** Determine the local maximum and minimum points of $f$.
**Solution:** Apply the First Derivative Test. Since the sign of $f'$ changes from negative to positive at $x = 0$, it follows that $x = 0$ is a local minimum point. Likewise, since the sign of $f'$ changes from positive to negative at $x = 4$, that point is a local maximum point.

**g)** Determine the points at which $f''(x)$ is undefined or equals 0.
**Solution:** From part c we see that $f''$ is defined for all $x$ but equals 0 at $x = 0$, 2, and 6.

**h)** Determine the intervals over which $f$ is concave up and down.
**Solution:** Use the concavity test; test the sign of the second derivative over the intervals determined by the points found in part g.

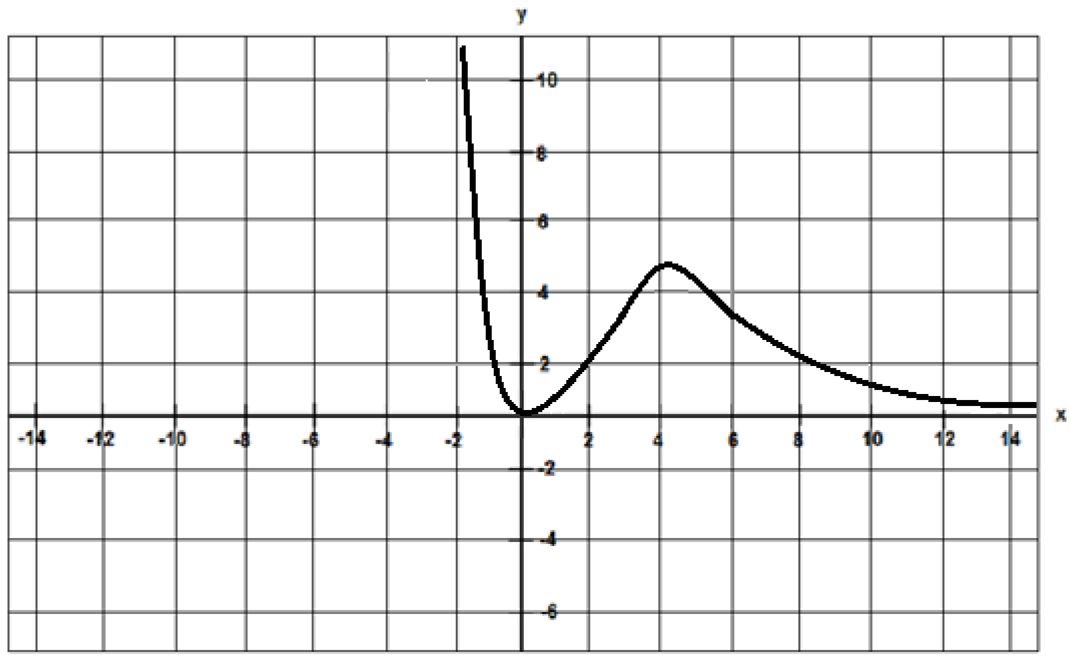
$$x < 0 \implies f''(-1) = 21e > 0; \qquad 0 < x < 2 \implies f''(1) = 5e^{-1} > 0$$

$$2 < x < 6 \implies f''(3) = -27e^{-3} < 0; \qquad x > 6 \implies f''(7) = 49 \cdot 4e{-7} > 0$$

Thus, $f$ is concave down on $(2, 6)$ and concave up on $(-\infty, 2)$ and $(6, \infty)$.

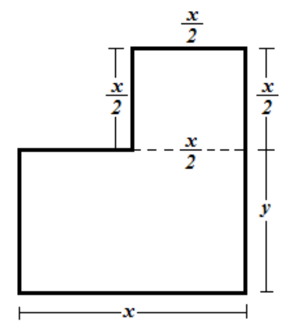
**i)** Find all inflection points of $f$.
**Solution:** The concavity changes at $x = 2$ and $x = 6$. So, the inflection points are $(2, f(2)) \approx (2, 2.17)$ and $(6, f(6)) \approx (6, 3.21)$.

**j)** Use this information - and any other easily obtainable information about $f(x)$ - to give an accurate sketch of the graph of $f$ on the axes below.
**Solution:** In addition to the information obtained in parts a - i, notice, also, that $f$ is a positive function: $x^4 e^{-x} > 0$ for every real number $x$. Thus, your graph cannot go below the $x$-axis and still get full credit.




The asymptote must be clearly indicated at the right. The local maximum at $x = 4$ has function value $f(4) \approx 4.68$, so it had to be close to that to get full credit.

**6.** A rancher wants to fence in a corral in the shape of a rectangle joined with a smaller square as in the figure below, where the side-length of the square portion is one-half the side of the rectangle it borders. There are only 1500 $m$ of fencing available. (NOTE: the fencing will only go along the *outer* boundary of the corral, which does *not* include the dotted line in the figure.) Work through the following steps to determine the dimensions of the corral that will yield the largest possible area given the stated conditions.




**a)** Label the relevant variables and clearly define the objective function (i.e. the quantity to be optimized) and the constraint equation in terms of your variables.

**Solution:** The quantity to be maximized is the area, which is the sum of the rectangular area and the square area. Noting the variables labeled above, we have $A = xy + (\frac{1}{2}x)^2 = xy + \frac{x^2}{4}$. The constraint is that there are only 1500 $m$ of fencing, which limits the perimeter: $1500 = P = 2y + x + \frac{x}{2} + 3(\frac{x}{2}) = 2y + 3x$.

**b)** Using the constraint, reduce the objective function to a function of a single variable, and determine an appropriate *closed, finite length* interval for the domain of this function as it pertains to this specific problem.

**Solution:** Solving $1500 = 2y + 3x$ for $y$ in terms of $x$, we have $y = 750 - \frac{3}{2}x$. Substitute this into $A = xy + \frac{x^2}{4}$ to obtain

$$A(x) = x\left(750 - \frac{3}{2}x\right) + \frac{x^2}{4} = 750x - \frac{6}{4}x^2 + \frac{1}{4}x^2 = 750x - \frac{5}{4}x^2.$$

From the constraint equation $1500 = 2y + 3x$, we see that $x$ cannot be any larger than 500, or else this equation could not hold (since $x$ and $y$ are both positive, being distances). Thus, an obvious domain for $A(x)$ is $0 \leq x \leq 500$; any number larger than 500 would also work.

**c)** Compute the derivative and critical points of the function you found in part b.

**Solution:** $A'(x) = 750 - \frac{5}{2}x$, which is defined for all $x$. $A'(x) = 0 \Rightarrow x = 2 \cdot 750/5 = 300$.

**d)** Determine the absolute maximum point of the function from part b on the interval you chose as its domain.

**Solution:** Since we've defined $A$ on a closed, finite length interval, we can use the standard extreme value method: testing $A(x)$ at the critical point and at the endpoints of $[0, 500]$.

$$A(0) = 0, \qquad A(500) = 62,500, \qquad A(300) = 112,500$$

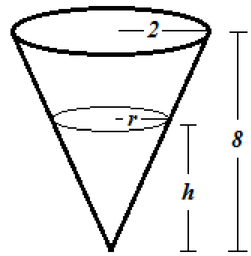
Thus, the absolute maximum of $A$ over $[0, 500]$ occurs at $x = 300$.

**e)** Determine the dimensions of the corral that yield the largest area subject to the given conditions.

**Solution:** Since the area is maximized when $x = 300$, we substitute this into $y = 750 - \frac{3}{2}x$ to obtain $y = 750 - 3 \cdot 150 = 750 - 450 = 300$ (i.e. the rectangluar area is actually square).

**7.** A water tank is in the shape of a cone with base radius 2 $m$ and height 8 $m$. Initially, the tank is full of water, but water begins flowing out of the bottom of the tank (where the vertex of the cone is) at a constant rate of 2 $m^3/min$. At what rate is the water level in the tank changing when the depth of the water is half the height of the cone? (The volume of a cone of base radius $r$ and height $h$ is $V = \frac{\pi}{3}r^2h$.)

**Solution:** You should be able to obtain the following picture.




We know that $\frac{dV}{dt} = -2$, and we want to compute $\frac{dh}{dt}$ when $h = 4$. Since we're not given any information about $r$, we can eliminate it from the volume formula using the usual similar triangle method ("flatten" the cone to obtain two concentric similar triangles): this yields $\frac{2}{r} = \frac{8}{h} \Rightarrow r = \frac{h}{4}$. Thus,

$$V = \frac{\pi}{3} \cdot \frac{h^2}{16} \cdot h = \frac{\pi}{48}h^3 \implies -2 = \frac{dV}{dt} = \frac{\pi}{16}h^2\frac{dh}{dt} \implies \frac{dh}{dt} = -\frac{32}{\pi h^2}.$$

When $h = 4$, we get $\frac{dh}{dt} = -\frac{2}{\pi}\ m/min \approx -0.637\ m/min$.

**8.** Compute (i.e *derive* - don't just write it down) the derivative of $f(x) = \cos^{-1} x$.

**Solution:** Let $y = \cos^{-1} x$, so that $cosy = x$. Implicitly differentiate to obtain

$$-(\sin y)\frac{dy}{dx} = 1 \implies \frac{dy}{dx} = -\frac{1}{\sin y}.$$

We know from the Pythagorean identity that $\sin^2 y + \cos^2 y = 1$, from which it follows that $\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$ (because $\cos y = x$).

Why can we take the positive square root? (This part was not required for the answer.) The function $f(x) = \cos^{-1}$ - by definition (see the textbook) - is positive on its natural domain, $[-1, 1]$, and has range $[0, \pi]$, which means that the variable $y$, above, is nonnegative and takes on values between 0 and $\pi$. Since the sine function is nonnegative on $[0, \pi]$, it follows that $\sin y \geq 0$. Hence, we can take the positive square root.

It now follows that

$$f'(x) = \frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}.$$

**BONUS: Choose *one* of the following to work. You will only receive credit for one of them. Note the different point values.**

**B1. (6 bonus pts)** State and prove the Mean Value Theorem. You may use Rolle's Theorem without proving it, but it must be clear where and how you use it.

**Solution:** *Mean Value Theorem - If a function, $f$, is continuous on a closed interval $[a, b]$ and differentiable on $(a, b)$, then there is a point, $c$, in $(a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

**Proof** Let $m = \frac{f(b)-f(a)}{b-a}$, and define a function $g(x) = m(x - a) + f(a)$. Then define a function $h(x) = f(x) - g(x)$. The sum/difference of two continuous functions is continuous, and the sum/difference of two differentiable functions is differentiable. Therefore, $h$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Moreover, we have

$$h(a) = f(a) - g(a) = f(a) - (m \cdot 0 + f(a)) = f(a) - f(a) = 0,$$

$$h(b) = f(b) - g(b) = f(b) - \big(m(b - a) + f(a)\big) = f(b) - \big(f(b) - f(a)\big) - f(a) = 0.$$

Thus, $h$ satisfies the conditions of Rolle's Theorem on $[a, b]$, which means that there is some point, $c$, in $(a, b)$ such that $h'(c) = 0$. But $h(x) = f(x) - g(x) \Rightarrow h'(x) = f'(x) - g'(x)$, which implies that $0 = h'(c) = f'(c) - g'(c) \Rightarrow f'(c) = g'(c)$. But $g$ is a linear function with slope $m$, so $g'(x) = m$ for *any* $x$, and, in particular, for $x = c$. Thus,

$$f'(c) = g'(c) = m = \frac{f(b) - f(a)}{b - a}.$$

**B2. (8 bonus pts)** Use the Mean Value Theorem to show that $e^x > x + 1$ for all positive real numbers, $x$. (This is a form of the well-known *Bernoulli inequality*, and its proof is much easier than it might look at first. HINT: $e^c > 1$ for every positive real number, $c$, and the Mean Value Theorem can be applied to $f(x) = e^x$ over any interval.)

**Solution:** Let $f(x) = e^x$. We know that this function is continuous and differentiable on the whole real line. Thus, the mean value theorem applies to this function over *any* interval, $[a, b]$. Let $x$ be any positive real number, and apply the mean value theorem to $f$ over the interval $[0, x]$. We can conclude that there is some point, $c$, in $(0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{e^x - e^0}{x} = \frac{e^x - 1}{x}.$$

Recalling that $f$ is its own derivative, we also know that $f'(c) = e^c$, and the above equation becomes

$$e^c = \frac{e^x - 1}{x}.$$

But $c > 0$ (because it lies between 0 and $x$), and, so, $e^c > 1$. Applying this to the previous equation, we see that

$$1 < e^c = \frac{e^x - 1}{x}.$$

Finally, since $x$ is positive, we can multiply through this last inequality by $x$ and it won't change the sign of the inequality. This gives us

$$x < e^x - 1 \implies x + 1 < e^x.$$

**B3. (10 bonus pts)** Consider the function $f(x) = (x+1)^{1/n}(x-1)^{(n-1)/n}$, where $n$ is an odd, positive integer greater than or equal to 3. Show that the graph of $f$ has both a cusp and a vertical tangent, and determine where they occur.

**Solution:** Differentiate $f$ using the produt rule, and them simplify to obtain

$$\begin{aligned}
f'(x) &= (x+1)^{\frac{1}{n}}\left[\frac{n-1}{n}(x-1)^{-\frac{1}{n}}\right] + (x-1)^{\frac{n-1}{n}}\left[\frac{1}{n}(x+1)^{\frac{1-n}{n}}\right] \\
&= \frac{(n-1)(x+1)^{\frac{1}{n}}}{n(x-1)^{\frac{1}{n}}} + \frac{(x-1)^{\frac{n-1}{n}}}{n(x+1)^{\frac{n-1}{n}}} \quad \textit{(now get a common denominator)} \\
&= \frac{(x+1)^{\frac{n-1}{n}}}{(x+1)^{\frac{n-1}{n}}} \cdot \frac{(n-1)(x+1)^{\frac{1}{n}}}{n(x-1)^{\frac{1}{n}}} + \frac{(x-1)^{\frac{1}{n}}}{(x-1)^{\frac{1}{n}}} \cdot \frac{(x-1)^{\frac{n-1}{n}}}{n(x+1)^{\frac{n-1}{n}}} \\
&= \frac{(n-1)(x+1)+(x-1)}{n(x-1)^{\frac{1}{n}}(x+1)^{\frac{n-1}{n}}} \\
&= \frac{nx+n-2}{n(x-1)^{\frac{1}{n}}(x+1)^{\frac{n-1}{n}}}
\end{aligned}$$

We can now see that $f'$ is undefined at $x = \pm 1$, and $f'(x) = 0$ at $x = \frac{2-n}{n}$. Note that $f$ is defined at $x = \pm 1$ but $f'$ is not. Thus, those are the candidates for the cusp and vertical tangent locations. So, we compute the one-sided limits of $f'$ at $\pm 1$. We find the limits at 1 first. Note that the numerator of $f'$ equals $2n-2$ at $x = 1$, which is positive because $n \geq 3$. In addition, the term $(x+1)^{(n-1)/n}$ equals $2^{(n-1)/n}$ at $x = 1$. It is the term $(x-1)^{1/n}$ in the denominator that equals 0 and determines the sign of the limit. As $x \to 1^+$, $x-1$ is a small *positive* number.

$$\lim_{x\to 1^+} \frac{nx+n-2}{n(x-1)^{\frac{1}{n}}(x+1)^{\frac{n-1}{n}}} = \frac{2n-2}{n \cdot 0^+ \cdot 2^{\frac{n-1}{n}}} = \frac{2n-2}{0^+} = \infty.$$

As $x \to 1^-$, $x-1$ is a small *negative* number. Moreover, note also that the $n^{th}$ root $(x-1)^{1/n}$ is defined even in this case because $n$ is odd. (If $n$ were even, this would not be the case.) Thus,

$$\lim_{x\to 1^-} \frac{nx+n-2}{n(x-1)^{\frac{1}{n}}(x+1)^{\frac{n-1}{n}}} = \frac{2n-2}{n \cdot 0^- \cdot 2^{\frac{n-1}{n}}} = \frac{2n-2}{0^-} = -\infty.$$

Hence, the graph of $f$ has a cusp at $x = 1$ (because the signs of the limits differ).

Now we compute the limits at $-1$. Note that the numerator equals $-2$ at $x = -1$. The term, $(x-1)^{1/n}$, in the denominator equals the $n^{th}$ root of $-2$, $(-2)^{1/n}$, which, again, is well-defined and negative because $n$ is odd. It's the term $(x+1)^{(n-1)/n}$ that goes to 0. However, this term is just $\left((x+1)^{1/n}\right)^{n-1}$, or the $(n-1)^{st}$ power of the $n^{th}$ root of $x+1$. Since $n$ is odd, $n-1$ is even. Even powers are always positive. Thus, even though this term will go to 0 as $x \to -1$ from either side, it will *always* be positive, from either side. Hence, we have

$$\lim_{x\to -1^\pm} \frac{nx+n-2}{n(x-1)^{\frac{1}{n}}(x+1)^{\frac{n-1}{n}}} = \frac{-2}{n \cdot (-2)^{\frac{1}{n}} \cdot 0^+} = \frac{-2}{-n2^{\frac{1}{n}} \cdot 0^+} = \infty.$$

Since the signs of the limits are the same, $f$ has a vertical tangent line at $x = -1$.