Math 1151 - Test 1: Solutions
1. Compute the following limits.
Solution

a) $$\lim_{x\to 2} \frac{\sqrt{x+14}}{2x^2} = \frac{\lim_{x\to 2}\sqrt{x+14}}{\lim_{x\to 2} 2x^2} = \frac{\sqrt{2+14}}{2\cdot 2^2} = \frac{\sqrt{16}}{8} = \frac{1}{2}$$

b) $$\lim_{x\to 2}\left(\frac{3}{2}(x-1)^2 e^{x^4-8x}\right) = \frac{3}{2}\left(\lim_{x\to 2}(x-1)^2\right)\left(\lim_{x\to 2} e^{x^4-8x}\right) = \frac{3}{2}(2-1)^2 e^{16-16} = \frac{3}{2}\cdot 1\cdot e^0 = \frac{3}{2}$$

c) $$\lim_{x\to 4^+} \frac{4x-1}{4-x}$$

The numerator approaches 15 - a positive number - as $x \to 4^+$. The denominator approaches 0 as $x \to 4^+$, but it does so from the *left*. For instance, if you take a number close to but greater than 4, say 4.001, and plug it into the denominator, you get -0.001. So the limit is $-\infty$.

d) $$\lim_{x\to -\infty} \frac{7x^6 - 4x^3 + 9x^2}{6x^7 - 3x^3 + 9x^2} = 0$$

This holds because the degree of the numerator is less than the degree of the denominator (a theorem we had from class), meaning that the limits at $\pm\infty$ are both 0. Essentially, the growth of the denominator for large x will dominate that of the numerator. You could have also derived this directly as follows.

$$\lim_{x\to -\infty} \frac{7x^6 - 4x^3 + 9x^2}{6x^7 - 3x^3 + 9x^2} = \lim_{x\to -\infty} \frac{x^7\left(\frac{7}{x} - \frac{4}{x^4}\frac{9}{x^5}\right)}{x^7\left(6 - \frac{3}{x^4} + \frac{9}{x^5}\right)} = \frac{\lim_{x\to -\infty}\left(\frac{7}{x} - \frac{4}{x^4}\frac{9}{x^5}\right)}{\lim_{x\to -\infty}\left(6 - \frac{3}{x^4} + \frac{9}{x^5}\right)} = \frac{0}{6} = 0$$

e) $$\lim_{x\to -1} \frac{\sqrt{x+2} - x}{x+1}$$

This limit does not exist, even as an infinite limit. This follows because

$$\lim_{x\to -1^+} \frac{\sqrt{x+2} - x}{x+1} = \frac{1+1}{0^+} = \infty,$$ where the 0^+ means "approaching 0 from the right," or positive

$$\lim_{x\to -1^-} \frac{\sqrt{x+2} - x}{x+1} = \frac{1+1}{0^-} = -\infty,$$ where the 0^- means "approaching 0 from the left," or negative.

Thus, since the one-sided limits are not equal, the limit - even in the infinite sense - does not exist.

2. ***Using the limit definition***, find the equation of the line tangent to the graph of the following function at the point corresponding to the given value of x.

$$f(x) = \frac{1}{\sqrt{2x+1}}, \quad x = 4$$

Solution 1 $f(4) = \frac{1}{\sqrt{9}} = \frac{1}{3}$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{2x+2h+1}} - \frac{1}{\sqrt{2x+1}}}{h} \\
&= \lim_{h\to 0} \frac{\frac{\sqrt{2x+1}}{\sqrt{2x+1}}\frac{1}{\sqrt{2x+2h+1}} - \frac{\sqrt{2x+2h+1}}{\sqrt{2x+2h+1}}\frac{1}{\sqrt{2x+1}}}{h} = \lim_{h\to 0} \frac{\sqrt{2x+1}-\sqrt{2x+2h+1}}{h\sqrt{2x+1}\sqrt{2x+2h+1}} \\
&= \lim_{h\to 0} \left[\frac{\sqrt{2x+1}-\sqrt{2x+2h+1}}{h\sqrt{2x+1}\sqrt{2x+2h+1}} \cdot \frac{\sqrt{2x+1}+\sqrt{2x+2h+1}}{\sqrt{2x+1}+\sqrt{2x+2h+1}}\right] \\
&= \lim_{h\to 0} \frac{2x+1-2x-2h-1}{h\sqrt{2x+1}\sqrt{2x+2h+1}\left(\sqrt{2x+1}+\sqrt{2x+2h+1}\right)} \\
&= \lim_{h\to 0} \frac{-2h}{h\sqrt{2x+1}\sqrt{2x+2h+1}\left(\sqrt{2x+1}+\sqrt{2x+2h+1}\right)} \\
&= \lim_{h\to 0} \frac{-2}{\sqrt{2x+1}\sqrt{2x+2h+1}\left(\sqrt{2x+1}+\sqrt{2x+2h+1}\right)} \\
&= \frac{-2}{\sqrt{2x+1}\sqrt{2x+1}\left(\sqrt{2x+1}+\sqrt{2x+1}\right)} = \frac{-2}{(2x+1)\cdot 2\sqrt{2x+1}} = \frac{-1}{(2x+1)^{3/2}}
\end{aligned}$$

$$\Longrightarrow f'(4) = \frac{-1}{9^{3/2}} = -\frac{1}{27} \implies \text{tangent line is } y - \frac{1}{3} = -\frac{1}{27}(x-4) \text{ or } y = -\frac{1}{27}x + \frac{13}{27}$$

Solution 2

$$\begin{aligned}
f'(4) &= \lim_{x\to 4} \frac{f(x)-f(4)}{x-4} = \lim_{x\to 4} \frac{\frac{1}{\sqrt{2x+1}} - \frac{1}{3}}{x-4} = \lim_{x\to 4} \frac{\frac{3-\sqrt{2x+1}}{3\sqrt{2x+1}}}{x-4} = \lim_{x\to 4} \frac{3-\sqrt{2x+1}}{3(x-4)\sqrt{2x+1}} \\
&= \lim_{x\to 4} \left[\frac{3-\sqrt{2x+1}}{3(x-4)\sqrt{2x+1}} \cdot \frac{3+\sqrt{2x+1}}{3+\sqrt{2x+1}}\right] = \lim_{x\to 4} \frac{9-2x-1}{3(x-4)\sqrt{2x+1}\left(3+\sqrt{2x+1}\right)} \\
&= \lim_{x\to 4} \frac{-2(x-4)}{3(x-4)\sqrt{2x+1}\left(3+\sqrt{2x+1}\right)} = \lim_{x\to 4} \frac{-2}{3\sqrt{2x+1}\left(3+\sqrt{2x+1}\right)} \\
&= \frac{-2}{3\cdot 3(3+3)} = -\frac{2}{54} = -\frac{1}{27} \quad \text{tangent line equation is same as above}
\end{aligned}$$

3. Determine all vertical and horizontal asymptotes (if any exist) for the following function.

$$f(x) = \frac{3x^5 + 15x^4 + 18x^3}{12x^5 - 24x^4 - 96x^3}$$

Solution

First, note that the degrees of the numerator and denominator are the same. Thus, $\lim_{x\to\pm\infty} f(x) = \frac{3}{12} = \frac{1}{4}$. This shows that $y = \frac{1}{4}$ is a horizontal asymptote for this function. Next, note that

$$f(x) = \frac{3x^3(x^2 + 5x + 6)}{12x^3(x^2 - 2x - 8)} = \frac{x^3(x+3)(x+2)}{4x^3(x+2)(x-4)}.$$

Thus, f is undefined at $x = 0$, -2, and 4; these are the candidates for possible vertical asymptotes. Vertical asymptotes are defined *not by the fact that terms cancel out in the function expression - which is simply a consequence of the form of rational functions but does not hold in general* - but by the fact that the limits at those points, from either side, are $\pm\infty$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x+3}{4(x-4)} = -\frac{3}{16} \;\rightarrow\; x = 0 \text{ is not a vertical asymptote}$$

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} \frac{x+3}{4(x-4)} = -\frac{1}{24} \;\rightarrow\; x = -2 \text{ is not a vertical asymptote}$$

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \frac{x+3}{4(x-4)} = \frac{7}{4\cdot 0^+} = \infty$$

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \frac{x+3}{4(x-4)} = \frac{7}{4\cdot 0^-} = -\infty \text{ (you only need one of these last two)}$$

Thus, $x = 4$ is the only vertical asymptote of f.

4. Compute the derivatives of the following functions. (You do not need to use the limit definition.)
Solutions

a) $f(x) = x^4 - 6x^3 + 2x^2 - 7x + 20$

$f'(x) = 4x^3 - 18x^2 + 4x - 7,$ by the summation/difference, constant multiple, and power rules

b) $u(t) = 4t^2e^t \cos t$

$$u'(t) = (4t^2e^t)\frac{d}{dt}(\cos t) + (\cos t)\frac{d}{dt}(4t^2e^t) = -4t^2e^t \sin t + (\cos t)(4t^2e^t + 8te^t) \text{ by product rule}$$

Or, if you applied the general product rule for any number of factors, you could also compute

$$u'(t) = 4t^2e^t\frac{d}{dt}(\cos t) + (4t^2 \cos t)\frac{d}{dt}(e^t) + (e^t \cos t)\frac{d}{dt}(4t^2) = -4t^2e^t \sin t + 4t^2e^t \cos t + 8te^t \cos t.$$

c) $r(v) = \dfrac{(2v+1)\sin v}{3v^3}$

$$\begin{aligned} r'(v) &= \frac{3v^3\frac{d}{dv}\Big((2v+1)\sin v\Big) - \Big((2v+1)\sin v\Big)\cdot\frac{d}{dv}(3v^3)}{9v^6} \\ &= \frac{3v^3\Big((2v+1)\cos v + 2\sin v\Big) - 9v^2(2v+1)\sin v}{9v^6} \\ &= \frac{v(2v+1)\cos v - (4v+3)\sin v}{3v^4} \end{aligned}$$

d) $y = \dfrac{3}{2x^{5/3}} - x^{3/5} - e^{-x} + \dfrac{1}{2}\cot x$

First rewrite: $y = \dfrac{3}{2}x^{-5/3} - x^{3/5} - e^{-x} + \dfrac{1}{2}\cot x$

$$\Longrightarrow y' = \frac{3}{2}\cdot\frac{-5}{3}x^{-8/3} - \frac{3}{5}x^{-2/3} - (-e^{-x}) + \frac{1}{2}(-\csc^2 x) = -\frac{5}{2x^{8/3}} - \frac{3}{5x^{2/3}} + e^{-x} - \frac{1}{2}\csc^2 x$$

5. Using the precise definition of a limit (i.e. the "epsilon-delta definition"), show that

$$\lim_{x\to 2}(x^3-1)=7.$$

Solution

Recall that to show $\lim_{x\to a} f(x)=L$, you must show the following: *given any* $\varepsilon>0$, *there is a* $\delta>0$ *such that* $|f(x)-L|<\varepsilon$ *whenever* $0<|x-a|<\delta$, *or - equivalently - such that* $-\varepsilon<f(x)-L<\varepsilon$ *whenever* $-\delta<x-a<\delta$, $x\neq a$. The way to do this is to work backwards from what you want to be true (i.e. $-\varepsilon<f(x)-L<\varepsilon$) until you obtain the inequality $-\delta<x-a<\delta$ for an appropriate choice of δ (which you must make). Then, you should be able to reverse your work to show that the δ you've chosen does, indeed, work.

So, let $\varepsilon>0$ be given. In this case, $f(x)=x^3-1$, $L=7$, and $a=2$. We want the following: $-\varepsilon<(x^3-1)-7<\varepsilon$.

$$-\varepsilon<x^3-8<\varepsilon \implies 8-\varepsilon<x^3<8+\varepsilon \implies \sqrt[3]{8-\varepsilon}<x<\sqrt[3]{8+\varepsilon}$$

This last step is valid and does not requre any further explanation or bounds on ε because cubed roots are defined for all real numbers, positive or negative, and the cubed root function is increasing (so if $a<b$, then $\sqrt[3]{a}<\sqrt[3]{b}$; this is why I chose a cubed root limit - to simplify the algebra).

NOTE: You canNOT split up the cubed root terms in the above inequality as $\sqrt[3]{8}\pm\sqrt[3]{\varepsilon}=2\pm\sqrt[3]{\varepsilon}$. It is NOT true that $\sqrt[3]{a\pm b}=\sqrt[3]{a}\pm\sqrt[3]{b}$.

Now, remember that we're working towards the inequality $-\delta<x-a<\delta$, where $a=2$ in this case. So, we want the middle of the above inequality to be $x-2$. Subtracting 2 from all three parts, we obtain

$$\sqrt[3]{8-\varepsilon}-2<x-2<\sqrt[3]{8+\varepsilon}-2.$$

Finally, we choose δ to be the smallest (in absolute value) of the two terms in the right and left sides of this last inequality. Clearly, $\sqrt[3]{8-\varepsilon}-2$ is less than $\sqrt[3]{8+\varepsilon}-2$ (because $8-\varepsilon<8+\varepsilon$). So, we choose $\delta=|\sqrt[3]{8-\varepsilon}-2|$.

I didn't require this for full credit, but you can check your work by starting with the inequality $\sqrt[3]{8-\varepsilon}-2<x-2<\sqrt[3]{8-\varepsilon}-2$, using the fact that $\sqrt[3]{8-\varepsilon}<\sqrt[3]{8+\varepsilon}$, and reversing your algebraic steps.

6. Consider the function

$$f(x)=\begin{cases}\frac{3x+4}{2x}, & x>2\\ \frac{1}{2}x+\frac{3}{2}, & x<2\\ 2, & x=2.\end{cases}$$

Solution

a) Show that f is not continuous at $x=2$.

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}\frac{3x+4}{x}=\frac{5}{2},\quad \lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}\left(\frac{1}{2}x+\frac{3}{2}\right)=\frac{5}{2}\implies \lim_{x\to 2} f(x)=\frac{5}{2}$$

But $f(2)=2\neq\frac{5}{2}$, so f is not continuous at 2 (i.e. $f(2)\neq\lim_{x\to 2} f(x)$).

b) Is this discontinuity essential or removable? Why? If it is removable, determine how you could redefine f to make it continuous at $x=2$.

It is removable because $\lim_{x\to 2} f(x)$ exists. Redefining $f(2)$ to be $\frac{5}{2}$ would make it continuous.

7. Show that $f'(0) = 0$ for the function below. Explicitly reference any theorems you use.

$$f(x) = \begin{cases} x^5 \cos\left(\frac{\pi^4}{2x^3}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Solution The product rule does not apply here. The product rule applies to a product $f(x) = g(x)h(x)$ only when g and h are differentiable at the point in question. The function $\cos(\pi^4/2x^3)$ is not only not differentiable at 0, it is not even continuous there. Moreover, the fact that $f(0) = 0$ does not mean that you can apply the "constant derivative rule" and claim that the derivative of 0 is 0. Derivatives (look back at the limit definition) depend on the function values at *and* nearby the point in question. The function f is *not* constant on any interval around 0. The correct way to proceed is as follows.

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{x^5 \cos\left(\frac{\pi^4}{2x^3}\right)}{x} = \lim_{x\to 0} x^4 \cos\left(\frac{\pi^4}{2x^3}\right)$$

The limit of $\cos(\pi^4/2x^3)$ at 0 does not exist, so you cannot apply the product rule for limits. You have to use the Squeezing Theorem. We know that $\cos\theta$ lies in between -1 and 1 for *any* real number θ. Thus,

$$-1 \leq \cos\left(\frac{\pi^4}{2x^3}\right) \leq 1 \implies -x^4 \leq x^4 \cos\left(\frac{\pi^4}{2x^3}\right) \leq x^4.$$

$$\lim_{x\to 0}(-x^4) = \lim_{x\to 0} x^4 = 0. \text{ So, by the Squeezing Theorem, } \lim_{x\to 0} x^4 \cos\left(\frac{\pi^4}{2x^3}\right) = 0$$

But, as we derived above, this limit is precisely $f'(0)$. Thus, $f'(0) = 0$.

8. If an archer shoots an arrow directly upward from the surface of the earth with an initial velocity of 50 m/s, its height above the ground at t seconds is given (approximately) by $h(t) = 50t - 5t^2$ m.

Solution

a) With what velocity will the arrow hit the ground?

$$h(t) = 0 \Rightarrow 50t - 5t^2 = 0 \Rightarrow 5t(10 - t) = 0 \Rightarrow t = 0, 10 \Rightarrow \text{arrow hits ground at 10 s}$$

$$v(t) = h'(t) = 50 - 10t \Rightarrow v'(10) = 50 - 100 = -50\ m/s$$

b) At what time during its flight does the arrow reach the peak of its ascent?

$$v(t) = 0 \Rightarrow 50 - 10t = 0 \Rightarrow t = 50/10 = 5\ s$$

d) What is the arrow's acceleration when it is halfway through its descent?

$$a(t) = h''(t) = v'(t) = -10 \Rightarrow a(t) \text{ is constant, the same for } all\ t.$$

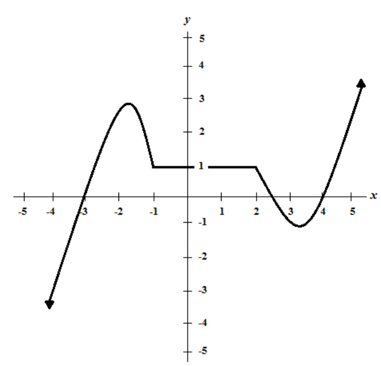
So, $a(t) = -10\ m/s^2$ for any t.

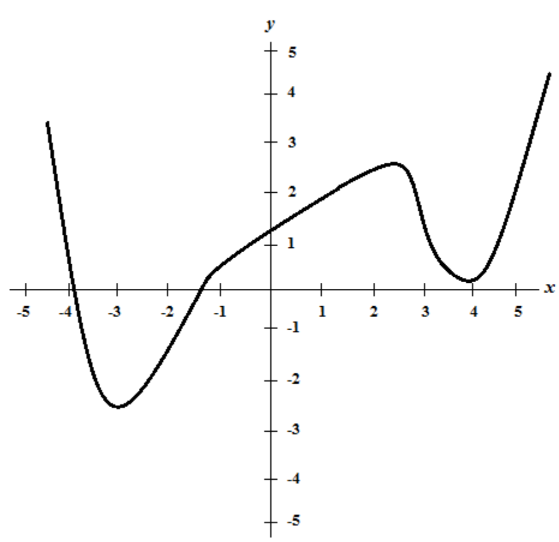
9. Show that the equation $\sqrt{2x^2 - 2x + 2} + x = 2$ has a solution between 0 and 1.
Solution

Let $f(x) = \sqrt{2x^2 - 2x + 2} + x - 2$. Then we want to show that the equation $f(x) = 0$ has a solution between 0 and 1. The function f is continuous on $[0, 1]$ because **1)** $\sqrt{2x^2 - 2x + 2}$ is the composition of a root function and a polynomial that is always positive (thus, the root is well-defined - and root functions are continuous where they are well-defined), **2)** $x - 2$ is a polynomial function, which is continuous everywhere, and **3)** f is the sum of these two functions, and the sum of continuous functions is also continuous.

Moreover, note that $f(0) = \sqrt{2} - 2 < 0$ and $f(1) = \sqrt{2} - 1 > 0$. Thus, $f(0) < 0 < f(1)$. Since 0 lies between the endpoint function values, and since f is continuous on $[0, 1]$, we can apply the Intermediate Value Theorem on this interval to conclude that there is some number c in $(0, 1)$ such that $f(c) = 0$. But this is precisely the equation we wanted to solve, so c is the desired solution.

10. The first figure below depicts the graph of the derivative, f', of a function, f. On the axes below it, give a sketch of what a graph of f might look like.





f must be decreasing for $x < -3$, $2.5 < x < 4$, increasing for $-3 < x < 2.5$, $x > 4$, have horizontal tangent lines at $x = -3$, 2.5, and 4. The flat, nonzero piece in the middle corresponds to a linear piece of f with a positive slope, but I counted that as a bonus point; it wasn't essential for the problem.

BONUS: The derivative of the inverse tangent function, $f(x) = \tan^{-1} x$, is $f'(x) = \frac{1}{1+x^2}$. Use the derivative (*not the graph*) to "prove" (a brief explanation and a computation or two will suffice) that the inverse tangent function has a horizontal asymptote.

Solution The actual proof of this result (and graphs do *not* constitute proofs) uses results that are not difficult but are beyond what we cover in this course. The intuitive proof I was looking for was very simple. The derivative of $f(x) = \tan^{-1} x$ is $f'(x) = \frac{1}{1+x^2}$. Horizontal asymptotes correspond to limits at $\pm\infty$. So, look first at the limit of the derivative at ∞ (the same reasoning holds for $-\infty$, also, but you were only asked to show that at least one horizontal asymptote existed).

$$\lim_{x\to\infty} \frac{1}{1+x^2} = 0 \text{ because } f' \text{ is rational with degree of the top} < \text{degree of the bottom}$$

Thus, $f'(x)$ goes to 0 as $x \to \infty$. But 0 derivatives correspond to horizontal tangent lines, or "flat spots" on the graph. Thus, the fact that the derivative approaches 0 as $x \to \infty$ means that the tangent lines to the graph of f are becoming closer and closer to being horizontal lines the further out on the real line (in the positive direction) you travel. Intuitively, the graph of f is "leveling off;" it is becoming flatter (but not precisely flat) because the tangent line slopes (i.e. $f'(x)$) are approaching 0. Thus, since the graph of f is leveling off, it must be approaching some fixed value. It can't continue increasing without bound, or else the derivative would not be going to 0. And since $\frac{1}{1+x^2} > 0$ for all x, the graph of f is never decreasing. Thus, f must approach some fixed real number at ∞, and this will indicate a horizontal asymptote.