CALCULUS II - 3450:222
TEST #3 SPRING 03

NAME ____________________

100 Points

Show all your work.

1. (5 pts) Does the following sequence converge or diverge? Show why.

$$a_n = 3 + \left(-\frac{1}{2}\right)^{n-1}$$

$$\lim_{n\to\infty}\left[3 + \left(-\frac{1}{2}\right)^{n-1}\right] = 3$$

converges to 3

5 Points

2. (13 pts) Determine whether the following series converges or diverges. If the series converges find its sum.

$$\sum_{n=1}^{\infty}\left[\frac{1}{2^n} - \frac{1}{3^{n-1}}\right] = \sum_{1}^{\infty}\left(\frac{1}{2}\right)^n - \sum_{1}^{\infty}\left(\frac{1}{3}\right)^{n-1}$$

$\sum_1^\infty \left(\frac{1}{2}\right)^n$ geometric $r = \frac{1}{2} < 1$ converges $\quad S = \frac{1/2}{1-1/2} = 1$

$\sum_1^\infty \left(\frac{1}{3}\right)^{n-1}$ geometric $r = 1/3 < 1$ converges $\quad S = \frac{1}{1-1/3} = \frac{3}{2}$

$\sum_1^\infty \left[\frac{1}{2^n} - \frac{1}{3^{n-1}}\right]$ converges $\quad S = 1 - \frac{3}{2} = -\frac{1}{2}$

13 Points

3. (9 pts) State whether the series converges absolutely, converges conditionally, or diverges.

$$\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{n}{4n^2-3}\right)$$

$\sum_1^\infty \left|(-1)^{n-1}\left(\frac{n}{4n^2-3}\right)\right| = \sum_1^\infty \frac{n}{4n^2-3} \geq \sum_1^\infty \frac{n}{4n^2} = \sum_1^\infty \frac{1}{4n}$ harmonic diverges

so $\sum_1^\infty \frac{n}{4n^2-3}$ diverges by C.T.

$\lim_{n\to\infty} \frac{n}{4n^2-3} = 0 \qquad f'(x) = \frac{4x^2-3-x(8x)}{(4x^2-3)^2} = \frac{-4x^2-3}{(4x^2-3)^2} < 0$ dec.

converges by AST

So $\sum_1^\infty (-1)^{n-1}\left(\frac{n}{4n^2-3}\right)$ conditionally convergent

9 Points

4. Determine whether the following series converge or diverge. State the test used and how it was used.

a. (8 pts) $\sum_{n=1}^{\infty} \frac{(3n-2)^2}{\sqrt{n^6+2n^4+1}}$ LCT with $\sum_1^{\infty} \frac{n^2}{\sqrt{n^6}} = \sum_1^{\infty} \frac{1}{n}$ harmonic diverges

$$\lim_{n\to\infty} \left[\frac{\frac{(3n-2)^2}{\sqrt{n^6+2n^4+1}}}{\frac{1}{n}} \right] = \lim_{n\to\infty} \left[\frac{(3n-2)^2}{\sqrt{n^6+2n^4+1}} \cdot n \right] = \lim_{n\to\infty} \left(\frac{9n^3}{n^3} \right) = 9 > 1$$

Both converge or diverge So $\sum_1^{\infty} \frac{(3n-2)^2}{\sqrt{n^6+2n^4+1}}$ diverges by LCT

b. (8 pts) $\sum_{n=1}^{\infty} n e^{-n}$ $= \sum_1^{\infty} \frac{n}{e^n}$

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{n+1}{e^{n+1}} \cdot \frac{e^n}{n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{(e)(n)} \right| = \frac{1}{e} < 1$$

$\sum_1^{\infty} \frac{n}{e^n}$ converges

c. (6 pts) $\sum_{n=1}^{\infty} [5-(0.2)^n]$ $= \sum_1^{\infty} \left[5 - \left(\frac{1}{5}\right)^n\right]$

$$\lim_{n\to\infty} \left(5 - \left(\tfrac{1}{5}\right)^n\right) = 5 \neq 0$$

diverges by Zero Test

d. (4 pts) $\sum_{n=1}^{\infty} \frac{2\sqrt{n}}{n^3}$ $= \sum_1^{\infty} \frac{2}{n^{5/2}} = 2 \cdot \sum_1^{\infty} \frac{1}{n^{5/2}}$

p-series $p = \frac{5}{2} > 1$

26 Points

converges

4. continued from previous page

e. (8 pts) $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

Ratio Test: $\lim_{n\to\infty} \left| \frac{3^{n+1} (n+1)^2}{(n+1)!} \cdot \frac{n!}{3^n n^2} \right|$

$= \lim_{n\to\infty} \left[\frac{3(n+1)^2}{(n+1) n^2} \right] = 0 < 1$

converges

8 Points

5. a. (3 pts) Write the Maclaurin series for e^x.

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$

b. (5 pts) Use part a to write a power series for e^{-x^2}

$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots$

c. (7 pts) Use part b to evaluate $\int_0^1 e^{-x^2} dx$. Note your answer will be an infinite series of constants.

$\int_0^1 e^{-x^2} dx = \int_0^1 \sum_0^\infty (-1)^n \frac{x^{2n}}{n!} dx$

$= \left[\sum_0^\infty (-1)^n \frac{x^{2n+1}}{n!(2n+1)} \right]_0^1$

$= \left[x - \frac{1}{3}x^3 + \frac{1}{5\cdot 2!} x^4 - \frac{1}{7\cdot 3!} x^7 + \cdots + (-1)^n \frac{x^{2n+1}}{n!(2n+1)} + \cdots \right]_0^1$

$= 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \cdots + (-1)^n \frac{1}{n!(2n+1)} + \cdots$

15 Points

6. (14 pts) Find the radius and interval of convergence of

$$\sum_{n=1}^{\infty} \frac{(x-4)^n}{n\,5^n}$$

Ratio Test: $\lim_{n\to\infty} \left| \frac{(x-4)^{n+1}}{(n+1)\,5^{n+1}} \cdot \frac{n\,5^n}{(x-4)^n} \right|$

$= \lim_{n\to\infty} \left(\frac{|x-4|\,n}{5(n+1)} \right) = \frac{|x-4|}{5}$

Converges: $\frac{|x-4|}{5} < 1$

$|x-4| < 5$

$-5 < x-4 < 5$

$-1 < x < 9$

test endpts:

$x=-1$: $\sum_{1}^{\infty} \frac{(-5)^n}{n\,5^n} = \sum_{1}^{\infty} (-1)^n \frac{1}{n}$ alternating haromic converges

$x=9$: $\sum_{1}^{\infty} \frac{5^n}{n\,5^n} = \sum_{1}^{\infty} \frac{1}{n}$ harmonic diverges

interval of convergence $= [-1, 9)$ radius $= 5$

14 Points

7. (10 pts) Find the first 4 terms in the Taylor Series for $f(x) = \ln(x)$ about $a = 2$.

$f'(x) = \frac{1}{x} = x^{-1}$ $f'(2) = \frac{1}{2}$

$f''(x) = -x^{-2} = -\frac{1}{x^2}$ $f''(2) = -\frac{1}{4}$

$f'''(x) = 2x^{-3} = \frac{2}{x^3}$ $f'''(2) = \frac{2}{8} = \frac{1}{4}$

$\ln x = \ln 2 + \frac{1}{2}(x-2) - \frac{1/4}{2!}(x-2)^2 + \frac{1/4}{3!}(x-2)^3$

$= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3$

10 Points