Honors CALC II Test 1A Grader: Stroup CAS 277

① Differentiate

a) $f(x)=5^x x^5 = e^{x\ln 5}x^5$

$f'(x)=5e^{x\ln 5}x^4+(\ln 5)e^{x\ln 5}x^5$

$=5\cdot 5^x x^4+5^x x^5 \ln 5$

b) $g(x)=\dfrac{e^{3x^2-1}}{(e^x)^2}$

$g'(x)=\dfrac{e^{2x}e^{3x^2-1}(6x)-e^{3x^2-1}e^{2x}(2)}{e^{4x}}=\dfrac{6xe^{3x^2-1}-2e^{3x^2-1}}{e^{2x}}$

c) $h(x)=\sec^{-1}(\sqrt{x}) \Rightarrow h'(x)=\dfrac{1}{2\sqrt{x}\sqrt{x}\sqrt{x-1}}$

$\boxed{\begin{array}{ll} u=\sqrt{x} & h=\sec^{-1}(u) \\ du=\dfrac{dx}{2\sqrt{x}} & dh=\dfrac{du}{u\sqrt{u^2-1}} \end{array}}$

$=\dfrac{1}{2x\sqrt{x-1}}$

d) $j(x)=e^{-5x}\cosh(2x) \Rightarrow j'(x)=-5e^{-5x}\cosh(2x)$

$\boxed{\begin{array}{ll} u=2x & v=\cosh(u) \\ du=2dx & dv=\sinh(u)du \end{array}}$

$+2e^{-5x}\sinh(2x)$

e) $K(x)=\log_2(\sqrt{x^2-4})=\ln(\sqrt{x^2-4})/\ln 2 \Rightarrow$

$\boxed{\begin{array}{lll} u=x^2-4 & v=\sqrt{u} & K=\ln(v) \\ du=2xdx & dv=\dfrac{du}{2\sqrt{u}} & dK=\dfrac{dv}{v} \end{array}}$

$K'(x)=\dfrac{1}{\ln 2}\cdot\dfrac{2x}{2(x^2-4)}$

$=\dfrac{x}{(\ln 2)(x^2-4)}$

② integrate

a) $\displaystyle\int \frac{dx}{x\sqrt{4-(\ln x)^2}} = \int \frac{dx}{x\sqrt{4\left(1-\left(\frac{\ln x}{2}\right)^2\right)}} = \int \frac{2\,du}{2\sqrt{1-u^2}} = \int \frac{du}{\sqrt{1-u^2}}$

$$\boxed{\begin{aligned} u &= \frac{\ln x}{2} \\ du &= \frac{dx}{2x} \\ dx &= 2x\,du \end{aligned}}$$

$= \sin^{-1}(u) + C$

$= \sin^{-1}\left(\frac{\ln x}{2}\right) + C$

b) $\displaystyle\int_0^{1/2} \frac{e^{2x}\,dx}{\sqrt{4-e^{2x}}} = \int_0^{1/2} \frac{e^{2x}\,dx}{2\sqrt{1-\frac{e^{2x}}{4}}} = \int \frac{du}{\sqrt{1-u}}$

$$\boxed{\begin{aligned} u &= \frac{e^{2x}}{4} \\ du &= \frac{e^{2x}}{2}\,dx \\ dx &= \frac{2\,du}{e^{2x}} \end{aligned}}$$

$= \displaystyle\int \frac{-dz}{\sqrt{z}} = \int -z^{-\frac{1}{2}}\,dz$

$$\boxed{\begin{aligned} z &= 1-u \\ dz &= -du \\ du &= -dz \end{aligned}}$$

$= -2z^{\frac{1}{2}}\Big|_{z(u(0))}^{z(u(\frac{1}{2}))} = -2\left(1-\frac{e^{2x}}{4}\right)^{\frac{1}{2}}\Big|_0^{\frac{1}{2}}$

$= -2\left[\left(1-\frac{e}{4}\right)^{\frac{1}{2}} - \left(1-\frac{1}{4}\right)^{\frac{1}{2}}\right]$

$= -2\left[\left(1-\frac{e}{4}\right)^{\frac{1}{2}} - \left(\frac{3}{4}\right)^{\frac{1}{2}}\right]$

2e) $\int \frac{\ln x}{x^2}\,dx$ Let $u = \ln x$, $dv = \frac{1}{x^2}dx$

$du = \frac{1}{x}dx$ $v = -\frac{1}{x}\,dx$

$$= -\frac{\ln x}{x} - \int -\frac{1}{x}\cdot\frac{1}{x}\,dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C$$

2f) $\int \sqrt{\cos x}\,\sin^3 x\,dx = \int \sqrt{\cos x}\,\sin^2 x\cdot\sin x\,dx$

$$= \int \sqrt{\cos x}\,(1-\cos^2 x)\sin x\,dx$$

Let $u = \cos x$, $du = -\sin x\,dx$

$$= -\int \sqrt{u}\,(1-u^2)\,du = \int u^{5/2} - u^{1/2}\,du$$

$$= \frac{2}{7}u^{7/2} - \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{7}(\cos x)^{7/2} - \frac{2}{3}(\cos x)^{3/2} + C$$

2g) $\int x^7 e^{-x^4}\,dx$, Let $t = -x^4$, $dt = -4x^3\,dx$

$$= \int x^4\cdot e^{-x^4}\cdot x^3\,dx = \int -t e^t \cdot -\frac{1}{4}\,dt$$

$$= \frac{1}{4}\int t e^t\,dt$$

Let $u = t$ $dv = e^t dt$

$du = dt$ $v = e^t$

$$= \frac{1}{4}\left[t e^t - \int e^t\,dt \right] = \frac{1}{4}(te^t - e^t) + C$$

$$= \frac{(t-1)}{4}e^t + C = \frac{(-x^4-1)}{4}e^{-x^4} + C$$

c) $\displaystyle\int \frac{t+1}{t^2+4}\,dt = \int \frac{t\,dt}{t^2+4} + \int \frac{t\,dt}{t^2+4}$

$u = t^2+4$
$du = 2t\,dt$
$dt = du/2t$

$\displaystyle = \frac{1}{2}\int \frac{du}{u} + \int \frac{dt}{4\left(\left(\frac{t}{2}\right)^2+1\right)}$

$\displaystyle = \frac{1}{2}\int \frac{du}{u} + \frac{1}{2}\int \frac{dz}{z^2+1}$

$z = \frac{t}{2}$
$dz = \frac{dt}{2}$
$dt = 2dz$

$\displaystyle = \frac{1}{2}\ln(t^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) + C$

d) $\displaystyle\int e^{2x}\tan(1+e^{2x})\,dt = \frac{1}{2}\int \tan(u)\,du$

$u = e^{2x}+1$
$du = 2e^{2x}dx$
$dx = \frac{du}{2e^{2x}}$

$\displaystyle = \frac{1}{2}\int \frac{\sin(u)}{\cos(u)}\,du$

$\displaystyle = \frac{-1}{2}\int \frac{dz}{z}$

$z = \cos u$
$dz = -\sin u\,du$
$du = \frac{dz}{-\sin u}$

$\displaystyle = -\frac{1}{2}\ln|z| + C$

$\displaystyle = -\frac{1}{2}\ln\left|\cos(e^{2x}+1)\right| + C$

(3) find limit

a) $\lim\limits_{x\to 0^+} x^{\sin x}$

$y = x^{\sin x}$

$\ln y = \sin x \ln x$

$\lim\limits_{x\to 0^+} \ln y = \lim\limits_{x\to 0^+} \sin x \ln x \overset{L'H}{=} \lim\limits_{x\to 0^+} \frac{\cos x}{x}$

$\overset{L'H}{=} \lim\limits_{x\to 0^+} -\sin x = 0$

$\lim\limits_{x\to 0^+} x^{\sin x} = \lim\limits_{x\to 0^+} e^{\sin x \ln x} = e^0 = 1$

b) $\lim\limits_{x\to\infty} x \ln\left(\frac{x+1}{x-1}\right) = \lim\limits_{x\to\infty} x(\ln(x+1) - \ln(x-1))$

$= \lim\limits_{x\to\infty} \frac{\ln(x+1) - \ln(x-1)}{1/x} \overset{L'H}{=} \lim\limits_{x\to\infty} \frac{\frac{1}{x+1} - \frac{1}{x-1}}{\frac{-1}{x^2}}$

$= \lim\limits_{x\to\infty} \frac{-x^2(x-1) + x^2(x+1)}{x^2-1} = \lim\limits_{x\to\infty} \frac{-x^3 + x^2 + x^3 + x^2}{x^2-1}$

$= \lim\limits_{x\to\infty} \frac{2x^2}{x^2-1} \overset{L'H}{=} \lim\limits_{x\to\infty} \frac{4x}{2x} = 2$

(4) given $f(x) = 4x + \cos(2x)$

a) show $f(x)$ has inverse on its whole domain

$$f'(x) = 4 - 2\sin(x)$$

$\Rightarrow f(x)$ is always increasing

$\Rightarrow f(x)$ is 1-1

$\Rightarrow f(x)$ has inverse on its domain

b) find slope of $f^{-1}(x)$ at $\left(5\pi, \frac{5\pi}{4}\right)$

$$f^{-1\prime}(5\pi) = \frac{1}{f'(f^{-1}(5\pi))} = \frac{1}{f'\left(\frac{5\pi}{4}\right)}$$

$$= \frac{1}{4 - 2\sin\left(\frac{5\pi}{4}\right)} = \frac{1}{4 + \sqrt{2}}$$

(5) differentiate

$$f(x) = (2x-5)^3 \sqrt[5]{\frac{x-1}{x+1}}$$

$$\ln y = 3\ln(2x-5) + \frac{1}{5}\ln(x-1) - \frac{1}{5}\ln(x+1)$$

$$y' = (2x-5)^3\left(\frac{x-1}{x+1}\right)^{\frac{1}{5}}\left[\frac{6}{2x-5} + \frac{1}{5(x-1)} + \frac{1}{5(x+1)}\right]$$