Honors Calculus II: Spring 2007
Exam 3

Name:____________________________________

1.[10pts.] Determine whether the improper integral $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$ converges or diverges. If it converges, find the value to which it converges.

$$= \lim_{b\to\infty} \int_0^b xe^{-x}dx + \lim_{a\to\infty}\int_a^0 xe^{-x^2} \quad ③$$

$$\int xe^{-x^2}dx \underset{\substack{u=-x^2\\du=-2xdx}}{=} \int -\tfrac{1}{2}e^u du = -\tfrac{1}{2}e^{-x^2} \quad ②$$

$$= \lim_{b\to\infty} -\tfrac{1}{2}e^{-x^2}\Big|_0^b + \lim_{a\to -a} -\tfrac{1}{2}e^{-x^2}\Big|_a^0 \quad ①$$

$$= \lim_{b\to\infty} -\tfrac{1}{2}e^{-b^2}+\tfrac{1}{2} + \lim_{a\to-\infty} -\tfrac{1}{2}+\tfrac{1}{2}e^{-a^2}$$

$$= 0 + \tfrac{1}{2} + -\tfrac{1}{2} + 0 \quad ②$$

$$= 0 \quad \text{Converges} \quad ②$$

2.[12pts.] Decide whether each of the following sequences converges or diverges. If a sequence converges, find its limit.

(a) $\{n^2e^{-n}\}$.

$$\lim_{n\to\infty} n^2e^{-n} = \lim_{n\to\infty}\frac{n^2}{e^n}, \frac{\infty}{\infty}$$

$$\overset{L'H}{=} \lim_{n\to\infty}\frac{2n}{e^n} \overset{L'H}{=} \lim_{n\to\infty}\frac{2}{e^n} = 0$$

② ②

Converges ①

(b) $\left\{\dfrac{\sin(2n)}{1+\sqrt{n}}\right\}$.

② ①

$$\lim_{n\to\infty}\left|\frac{\sin 2n}{1+\sqrt{n}}\right| \le \lim_{n\to\infty}\frac{1}{1+\sqrt{n}} = 0$$

Converge ①

(c) $\{0,1,0,0,1,0,0,0,1,0,0,0,0,1,\ldots\}$.

diverge, oscillates

2 2

3.[32pts.] Decide whether each of the following series converges absolutely, converges conditionally, or diverges.

(a) $\sum_{n=0}^{\infty} \frac{2n^2+3}{3n^4+4}$.

8

L.C.T. w/ $\Sigma \frac{1}{n^2}$, convergent p-series (3)

$\lim_{n\to\infty} \frac{\frac{2n^2+4}{3n^4+4}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2n^4+4n^2}{3n^4+4}$ (3)

$= 3/4$ (1) not 0 or ∞

So by L.C.T. $\sum_{n=0}^{\infty} \frac{2n^2+3}{3n^4+4}$

Converges (1)

(b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2+3}{5n^2}$.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \frac{n^2+3}{5n^2} \neq \frac{+1}{4}$

6

So diverge (2)

(c) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+9}$.

assoc. pos series

$\sum_{n=1}^{\infty} \frac{n}{n^2+9}$ (2)

L.C.T. w/ $\sum_{n=1}^{\infty} \frac{1}{n}$, divergent (2)

harmonic series.

$\lim_{n\to\infty} \frac{\frac{n}{n^2+9}}{\frac{1}{n}} = 1$

So $\sum_{n=1}^{\infty} \frac{n}{n^2+9}$ does not converge (1)

absolutely.

By A.S.T. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+9}$

(1) Conditionally Converges since

$\lim_{n\to\infty} \frac{n}{n^2+9} = 0$ & $\left[\frac{n}{n^2+9}\right]' = \frac{-n^2+9}{(n^2+9)^2} < 0$ for $n > 3$.

(1) (1)

(d) $\sum_{n=1}^{\infty} \frac{\cos(n/2)}{n^2}$.

assoc pos series $\sum_{n=1}^{\infty} \frac{|\cos \frac{n}{2}|}{n^2}$

$\left|\frac{\cos(\frac{n}{2})}{n^2}\right| < \frac{1}{n^2}$ (3)

Since $\Sigma \frac{1}{n^2}$ is a convergent p-series,

By B.C.T., (3) $\sum_{n=1}^{\infty} \frac{\cos \frac{n}{2}}{n^2}$ converges

absolutely (2)

4.[15pts.] (a) Use the binomial series to expand $f(x) = (1+2x)^{3/4}$ as a Maclaurin series and to find the first three Taylor polynomials T_1, T_2, and T_3. (b) Use Taylor's Inequality (Taylor's Remainder Theorem) to estimate the error when using T_2 to approximate $f(x)$ at $x = .1$.

a) $(1+2x)^{3/4} = \sum_{n=0}^{\infty} \binom{3/4}{n}(2x)^n$

$T_1 = \binom{3/4}{0} + \binom{3/4}{1}x = 1 + 3/4\,x$

$T_2 = \binom{3/4}{0} + \binom{3/4}{1}x + \binom{3/4}{2}x^2 = 1 + 3/4\,x + \frac{3/4(\frac{3}{4}-1)}{2!}x^2$

$T_3 = \binom{3/4}{0} + \binom{3/4}{1}x + \binom{3/4}{2}x^2 + \binom{3/4}{3}x^3 = 1 + 3/4\,x + \frac{3/4(3/4-1)}{2!}x^2 + \frac{3/4(3/4-1)(3/4-2)}{3!}x^3$

$T_1 = 1 + 3/4$, $T_2 = 1 + 3/4\,x - \frac{3}{32}x^2$, $T_3 = 1 + 3/4\,x - \frac{3}{32}x^2 + \frac{5}{128}x^3$

(2) (2) (2) (2)

b) $f(.1) = T_2(.1) + R_2(.1)$

$R_2(.1) = \frac{f^{(3)}(\xi)(.1)^3}{3!}$ (3)

$\le \frac{\frac{15}{8}(.1)^3}{6} = 3.125E{-4}$ (1)

$f' = \frac{3}{4}(1+2x)^{-1/4}\cdot 2 = \frac{3}{2}(1+2x)^{-1/4}$

$f'' = \frac{3}{2}\cdot(-\frac{1}{4})(2)(1+2x)^{-5/4} = \frac{3}{4}(1+2x)^{-5/4}$

$f''' = \frac{3}{4}(-5/4)\cdot 2\,(1+2x)^{-9/4} = -\frac{15}{8}(1+2x)^{-9/4}$ (3)

$f'''(\xi) = \frac{-15}{8(1+2x)^{9/4}} < \frac{15}{8}$

5.[13pts.] Use a power series to approximate $\int_0^{.5} \frac{1}{1+x^6}\,dx$ with an error less than 1×10^{-7}.

(2) $\sum_{n=0}^{\infty} \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$

(2) $\sum_{n=0}^{\infty} \frac{1}{1+x^6} = \sum_{n=0}^{\infty} (-1)^n x^{6n}$

(2)

$\int_0^{.5} \frac{1}{1+x^6}dx = \int_0^{.5} \sum_{n=0}^{\infty} (-1)^n x^{6n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{6n+1} x^{6n+1}\Big|_0^{1/2}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2})^{6n+1}}{6n+1}$ (1)

$= \boxed{\frac{1}{2} - \frac{1}{7}(\frac{1}{2})^7 + \frac{1}{13}(\frac{1}{2})^{13} - \frac{1}{19}(\frac{1}{2})^{19}} + \frac{1}{25}(\frac{1}{2})^{25}$

1.003E-7 $\quad$ 1.19×10^{-9}

(4) $\quad$ (2)

$= .4983932$

6.[13pts.] Find the center, the interval, and the radius of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{2^n(x-3)^n}{\sqrt{n+3}}.$$

$$\lim_{n\to\infty} \left| \frac{\frac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+4}}}{\frac{2^n(x-3)^n}{\sqrt{n+3}}} \right| = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \left| \frac{(x-3)^{n+1}}{(x-3)^n} \right| \frac{\sqrt{n+3}}{\sqrt{n+4}} = 2|x-3|<1$$ (4)

$2|x-3|<1$ or $(x-3)<\frac{1}{2}$ $\Rightarrow$ $-\frac{1}{2}<x-3<\frac{1}{2}$ $\Rightarrow$ $\frac{5}{2}<x<\frac{7}{2}$? ?

@ $x=5/2$ $\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+3}}$ (2) Alt. Series test $\lim_{n\to\infty}\frac{1}{\sqrt{n+3}}=0$ $\left[\frac{1}{\sqrt{n+3}}\right]'=\frac{-\frac{1}{2}}{(n+3)^{3/2}}<0$ Converges.

@ $7/2$ $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+3}}$ diverges by L.C.T w/ divergent p series

(1)

(2) Radius $\frac{1}{2}$

(2) Center 3

(2) interval $[\frac{5}{2}, 7/2)$

7.[5pts.] Find the sum of the series $\sum_{n=1}^{\infty} 3\frac{2^{n-1}}{3^n}$. $= \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1}$

$= 1+\frac{2}{3}+\left(\frac{2}{3}\right)^2$

(2) $a=1$ $r=2/3$ (2)

$S=\frac{1}{1-2/3}=3$

(1)