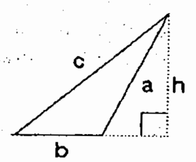
Knowledge of formulas for length, perimeter and area of plane regions as well as for volume and surface area of solids are extremely important in Calculus. In the formulas below, distances are indicated by a, b, c, h, r and s; the angle, θ, is measured in radians; areas of plane regions are indicated by A, surface areas of solids by SA and volumes of solids by V.

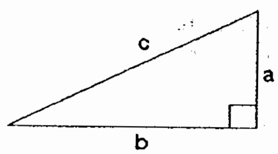
General Triangle



$P = a + b + c$

$A = \frac{1}{2}bh$

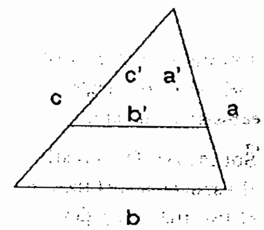
Right Triangle



$a^2 + b^2 = c^2$

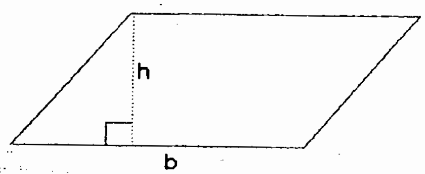
$A = \frac{1}{2}ab$

Similar Triangles



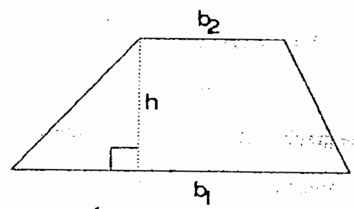
$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c}$

Parallelogram



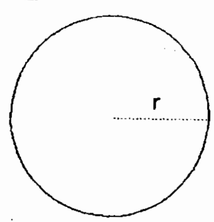
$A = bh$

Trapezoid



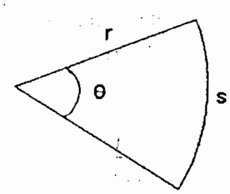
$A = \frac{1}{2}(b_1 + b_2)h$

Circle



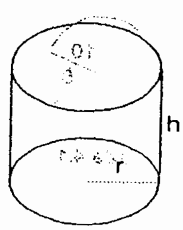
$P = 2\pi r, \ A = \pi r^2$

Circular Sector



$s = r\theta, \ A = \frac{1}{2}r^2\theta$

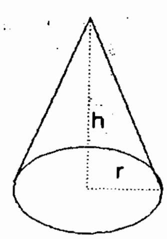
Right Circular Cylinder



$V = \pi r^2 h$

$SA = 2\pi rh + 2\pi r^2$

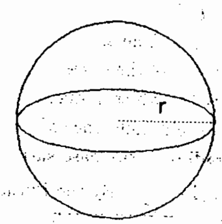
Right Circular Cone



$V = \frac{1}{3}\pi r^2 h$

$SA = \pi r^2 + \pi r\sqrt{r^2 + h^2}$

Sphere



$V = \frac{4}{3}\pi r^3$

$SA = 4\pi r^2$

Examples of functions related to geometry.

EXAMPLE 1: A five foot tall student walks away from a 15 foot high lamp post. Express the length of her shadow as a function of distance from the lamp post.

SOLUTION: The situation is illustrated in Figure 1. Let x be her distance from the lamp post and L the length of her shadow. By similar triangles

$$\frac{x+L}{L} = \frac{15}{5} \implies x + L = 3L$$

$$\implies L(x) = \frac{1}{2}x,\ x \geq 0$$

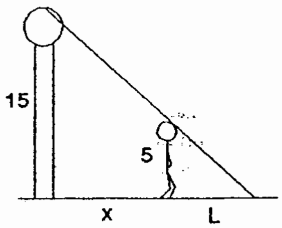


Figure 1

EXAMPLE 2: A rain-gutter is to be made from a 10 in. wide sheet of metal by bending the sheet x inches from each edge (the angle formed should be larger than a right angle) to form a trough that is 3 in. high. Express the cross-sectional area of this trough as a function of x. Draw a complete graph of the problem situation.

SOLUTION: The situation is illustrated in Figure 2.1. The area is the sum of area of the rectangle of sides $3 \times (10-2x)$ and the areas of two right triangles of height 3 and hypotenuse x. Notice that $x \leq 5$ is required.
By the Pythagorean Theorem, the other side of the triangles is of length $\sqrt{x^2-3^2}$, so the area of each triangle is $A_\Delta = \frac{1}{2}3\sqrt{x^2-3^2}$, $x \geq 3$.
The cross-section has area

$$A(x) = 3(10-2x) + 2\frac{1}{2}3\sqrt{x^2-3^2}$$

$$\implies A(x) = 3(10-2x+\sqrt{x^2-9}),\ 3 \leq x \leq 5$$

The graph of $A(x)$ is drawn in Figure 2.2.

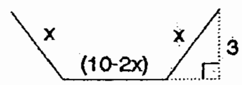
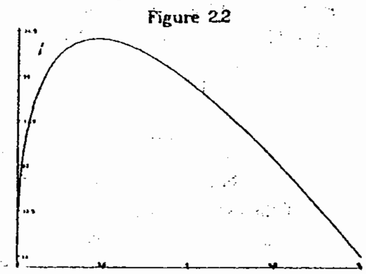


Figure 2.1

Figure 2.2



EXAMPLE 3: A 400 meter oval track is to be built with two equal length straight sides connected by a semi-circle at each end. Write the area inside the track as a function of the length of the straight side. Draw a complete graph of the problem situation.

SOLUTION: The situation is illustrated in Figure 3.1. The area is the sum of the area of the rectangle of sides $x \times 2r$ and the areas of the two semi-circles, each of radius r. Notice that $0 \leq x \leq 200$. The perimeter of the track is $P = 2x + 2(\pi r) = 400$, so the variable r is determined in terms of x as $r = \frac{1}{\pi}(200-x)$. Thus the total area is

$$A(x) = x(2r) + 2(\frac{1}{2}\pi r^2) = \frac{2x}{\pi}(200-x) + \frac{1}{\pi}(200-x)^2$$

$$\implies A(x) = \frac{1}{\pi}(200-x)(200+x),\ 0 \leq x \leq 200$$

This quadratic is graphed over the problem domain in Figure 3.2.

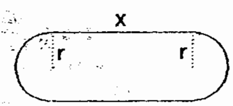
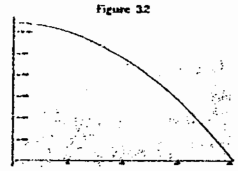


Figure 3.1

Figure 3.2



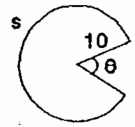
EXAMPLE 4: A drinking cup is made by removing a circular sector of angle θ (radians) from a circle of radius 10 cm and fastening the edges together (see Figure 4.1 below). Find the volume of the conical cup as a function of the angle of the sector which was removed. Draw a complete graph of the problem situation.

SOLUTION: The situation is illustrated in Figure 4.1. The length of the part of the circle that remains is $s = 10(2\pi - \theta)$, $0 \leq \theta \leq 2\pi$. The sector which remains is folded into the cup shown in Figure 4.2. The circumference of the base of the cone is also $s = 10(2\pi - \theta)$. The radius of the base satisfies the equation $2\pi r = 10(2\pi - \theta)$ so $r = 10(1 - \frac{\theta}{2\pi})$. By the Pythagorean Theorem, $h = \sqrt{10^2 - r^2}$, so $h = 10\sqrt{1 - \left(1 - \frac{\theta}{2\pi}\right)^2}$. Substituting these values for r and h in the formula for the volume of the cone gives

$$V(\theta) = \frac{1}{3}\pi r^2 h$$

$$\implies V(\theta) = \frac{10^3\pi}{3}\left(1 - \frac{\theta}{2\pi}\right)^2 \sqrt{1 - \left(1 - \frac{\theta}{2\pi}\right)^2},\ 0 \leq \theta \leq 2\pi$$

The graph of is drawn in Figure 4.3.



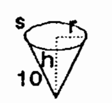
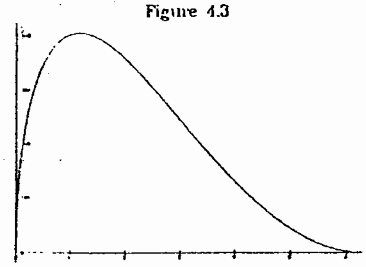


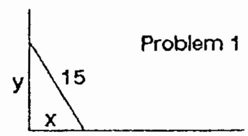
Figure 4.1 Figure 4.2

Figure 4.3

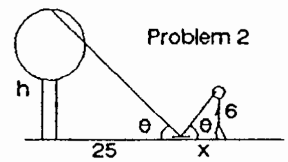


Problems on functions related to geometry

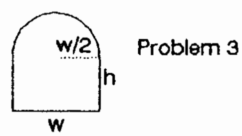
1. A 15 foot ladder leans against a wall. A painter is at the top of the ladder when the bottom begins to slip away from the wall. Determine the distance of the top of the ladder from the ground when the bottom is x feet from the base of the wall.



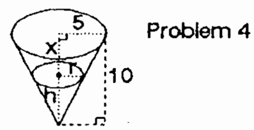
2. One way of measuring the height of a tree is to place a mirror on the ground 25 feet from the base of the tree, then back away from the tree until you can just see the reflection of the top of the tree in the mirror. You can calculate the height of the tree as a function of distance from the mirror. Determine how high your eyes are above the ground and find the function which describes height in terms of distance from the mirror. (The physical principle used here is that *Angle of incidence equals angle of reflection.*)



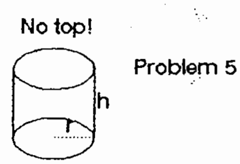
3. A *Roman Window* has the shape of a rectangle with a semi-circle above. The diameter of the semi-circle and the top of the rectangle are the same line. Find the function which represents the perimeter of such a window as a function of width if the window is to have a fixed area of 30 sq. ft. Draw a complete graph of the problem situation.



4. The conical tank at right is being filled with water. What is the volume of water in the tank when the distance from the top of the tank to the surface of the water is x feet? Draw a complete graph of the problem situation.



5. A cylindrical bucket (with bottom but no top) is to have a volume capacity of 1 cubic meter. Find function which describes the surface area of the bucket as a function of the radius of the cylinder (no top). Draw a complete graph of the problem situation.



Answers to problems

1. $y = \sqrt{225 - x^2}$, $0 \le x \le 15$ **2.** $h(x) = \dfrac{150}{x}$, $x > 0$ (assumes my eyes are 6 feet high). **3.** $P(w) = (1 - \frac{\pi}{3})w + \dfrac{60}{w}$, $0 < w \le \sqrt{\dfrac{240}{\pi}}$ Graph below. **4.** $V(x) = \dfrac{\pi}{12}(10 - x)^3$, $0 \le x \le 10$. Graph below. **5.** $SA(r) = \dfrac{2}{r} + \pi r^2$, $0 < r$. Graph below.