Calc I - 3450:221
FINAL

NAME Key
ROW

Show **ALL** your work.

200 Points

1. Find the average value of the function $f(x) = x^2 - 4x + 5$ over $x \in [-1, 2]$.

10 Points

2 2 1

$$\frac{1}{2-(-1)}\int_{-1}^{2}\left(x^2-4x+5\right)dx$$

$$= \frac{1}{3}\left(\frac{x^3}{3}-2x^2+5x\right)\Big|_{-1}^{2}$$ 3

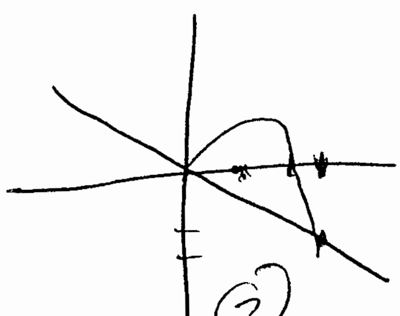
$$= \frac{1}{3}\left[\left(\frac{8}{3}-8+10\right)-\left(\frac{-1}{3}-2-5\right)\right] = \frac{1}{3}(3+9) = \frac{1}{3}\cdot 12 = 4$$

2

2. Find the area of the region bounded by $y = -x$ and $y = x - x^2$.

10 Points

$$-x = x - x^2 \Rightarrow x^2 - 2x = 0 \Rightarrow x = 0, 2$$



2 4

$$A = \int_0^2\left[(x-x^2)-(-x)\right]dx = \int_0^2\left(2x-x^2\right)dx$$

20 Points

$$= \left(x^2-\frac{x^3}{3}\right)\Big|_0^2 = 4-\frac{8}{3} = \frac{4}{3}$$

2 2

3. Find $\frac{dy}{dx}$ if $y = \sqrt{\frac{5x^2 - 7}{9 - 2x^3}}$. You do not have to simplify your answer.

$$y' = \frac{1}{2}\left(\frac{5x^2-7}{9-2x^3}\right)^{-1/2} \cdot \frac{(9-2x^3)\cdot 10x - (5x^2-7)(-6x^2)}{(9-2x^3)^2}$$

7 Points

4. Find y'' if $y = \cos(3x^2)$.

$$y' = -\sin(3x^2)\cdot 6x$$

$$y'' = -\cos(3x^2)\cdot 6x \cdot 6x - 6\cdot \sin(3x^2)$$

8 Points

5. Find the equation of the line normal to $5x^3 - 2x^2y^2 + 4y^3 - 29 = 0$ at the point (1, 2).

$$15x^2 - 4xy^2 - 4x^2 y y' + 12y^2 y' = 0$$

$$\Rightarrow y' = \frac{-15x^2 + 4xy^2}{-4x^2y + 12y^2} = \frac{-15+16}{-8+48} = \frac{1}{40}$$

Slope normal $= -40$

$$\Rightarrow \frac{y-2}{x-1} = -40$$

10 Points

25 Points

6. Consider the region bounded by the graphs of y = −(x + 1) and x = −(y − 1)^2 for y ≥ 1. SET UP THE INTEGRAL(S) needed to find the volume of the solid of revolution formed by revolving this region about the

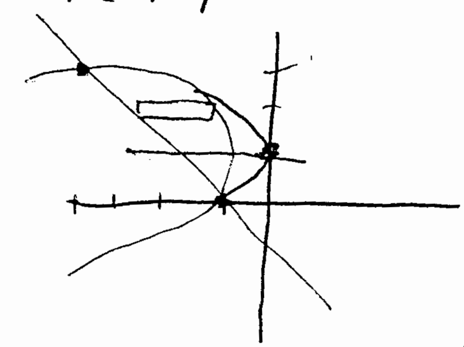
a) x-axis (Set up the integral(s) for integration with respect to y.)

$x = -y^2 + 2y \cancel{-1} = \cancel{-1} - y \Rightarrow -y^2 + 3y = 0 \Rightarrow y = 0, 3$

$x = -1, -4$

10 Points

$x = -1 - y$



shells

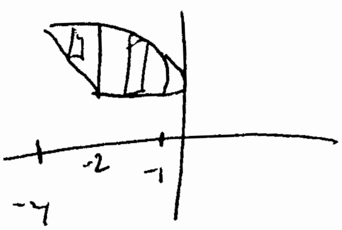
$$V = \int_1^3 2\pi \cdot y \cdot \left[-(y-1)^2 - (-1-y)\right] dy$$

2 2 2 2 2

$x = -(y-1)^2 \Rightarrow -x = (y-1)^2 \Rightarrow y - 1 = \sqrt{-x} \Rightarrow y = 1 + \sqrt{-x}$

b) x-axis (Set up the integral(s) for integration with respect to x.)

10 Points



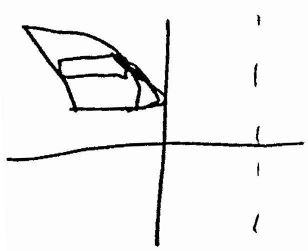
washers

$$V = \int_{-4}^{-2} \pi \left[(1+\sqrt{-x})^2 - (-1-x)^2\right] dx + \int_{-2}^{0} \pi \left[(1+\sqrt{-x})^2 - 1^2\right] dx$$

2 1 3 2 1 3

c) line x = 2 (Indicate the method used.)

10 Points



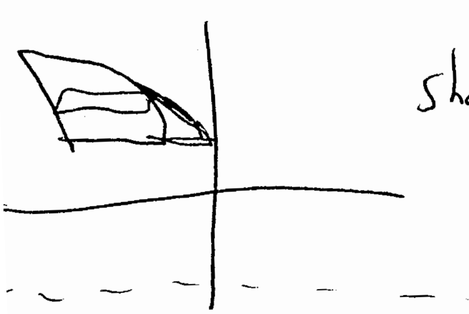
washers

$$V = \int_1^3 \pi \left[(+1+y+2)^2 - (2 + (y-1)^2)^2\right] dy$$

3 3 2 2

d) line y = -2 (Indicate the method used.)

10 Points
40 Points



shells

$$V = \int_1^3 2\pi (2+y) \left[-(y-1)^2 - (-1-y)\right] dy$$

2 2 2 2 2

12½ min

7. Evaluate $\int_0^1 \frac{x^2 - 3x + 2}{\sqrt{x}}\, dx$.

10 Points

$$\int_0^1 \left(x^{3/2} - 3x^{1/2} + 2x^{-1/2}\right) dx \quad 5$$

$$= x^{5/2}\cdot \tfrac{2}{5} - 3x^{3/2}\cdot \tfrac{2}{3} + 4x^{1/2} \Big|_0^1$$

1 1 1

$$= \tfrac{2}{5} - 2 + 4 = 2\tfrac{2}{5} = \tfrac{12}{5}$$

2

8. Evaluate $\int_{-2}^{6} (3 - 7x^5)^3\, dx + \int_{6}^{-2} (3 - 7x^5)^3\, dx$.

10 Points

0

10

9. Evaluate $\int \frac{2 - 6\sin(3x)}{[x + \cos(3x)]^2}\, dx$.

10 Points

let $u = x + \cos 3x$ 3

$du = (1 - 3\sin 3x)\, dx$ 2

$$\int \frac{2\, du}{u^2} = -2u^{-1} + C$$

2 1 1

$$= -2(x + \cos 3x)^{-1} + C$$

1

30 Points

10. Evaluate $\int 3x \sec^2(5x^2)\, dx$.

10 Points

let $u = 5x^2$ 2

$du = 10x\, dx$ 2

$$\int \frac{3}{10} \sec^2 u \, du = \frac{3}{10} \tan u + C$$

2 2 1

$$= \frac{3}{10} \tan(5x^2) + C$$

1

11. Evaluate $\int_1^2 x^2\sqrt{x-1}\, dx$.

10 Points

let $u = x - 1 \Rightarrow x = 1 + u$ $x = 1 \Rightarrow u = 0$

$du = dx$ 2 $x = 2 \Rightarrow u = 1$

2

$$\int_0^1 (1+u)^2 u^{1/2}\, du$$

2

$$= \int_0^1 (1 + 2u + u^2) u^{1/2}\, du$$

$$= \int_0^1 \left[u^{1/2} + 2u^{3/2} + u^{5/2}\right] du$$

1

$$= u^{3/2} \cdot \tfrac{2}{3} + 2u^{5/2} \cdot \tfrac{2}{5} + u^{7/2} \cdot \tfrac{2}{7} \Big|_0^1$$ 2

20 Points

$$= \frac{2}{3} + \frac{4}{5} + \frac{2}{7} = \frac{70 + 84 + 30}{105} = \frac{184}{105}$$ 1

12. Evaluate $\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^9 \frac{1}{n}$.

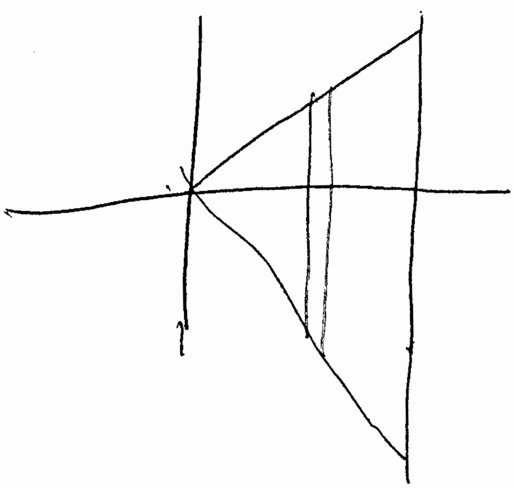
10 Points

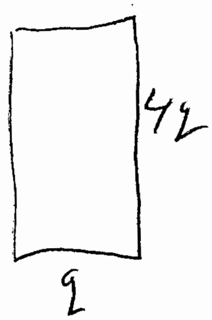
$$\int_0^1 x^9\,dx = \frac{x^{10}}{10}\Big|_0^1 = 1/10$$

4 4 1 1

13. A solid has as its base the region in the plane bounded by $y = x$, $x = 4$ and $y = -2x$. Each cross section perpendicular to the plane and parallel to the y–axis is a rectangle with height four times the length of the base which lies in the plane. Find the volume of this solid.

10 Points





$q = x - (-2x) = 3x$

$A = 4q^2 = 4\cdot(3x)^2 = 36x^2$

$$\int_0^4 36x^2\,dx = \frac{36}{3}x^3\Big|_0^4 = 12x^3\Big|_0^4 = 12\cdot 64$$

$= 768$

2 5 2 1

64
×2
128
64

20 Points

14. Answer all of the questions below for $f(x) = \dfrac{x^3 - 1}{x^3 + 1}$.

a) Find any vertical, horizontal, and slant asymptotes. Show all limits used.

7 Points

$\lim_{x \to \pm\infty} f(x) = 1 \Rightarrow y = 1$ horizontal

3

$\lim_{x \to -1^+} \dfrac{x^3-1}{x^3+1} = -\infty$ (-2) (+0)

2

$\lim_{x \to -1^-} \dfrac{x^3-1}{x^3+1} = +\infty$ (-2) (-0)

2

$X = -1$ vertical

b) Find the intervals where f(x) is increasing/decreasing and identify any extrema.

8 Points

$$f' = \frac{(x^3+1)3x^2 - (x^3-1)\cdot 3x^2}{(x^3+1)^2} = \frac{6x^2}{(x^3+1)^2} > 0 \text{ always}$$

4

so always inc

4

no extrema

c) Find the intervals where f(x) is concave up/down and identify any points of inflection.

10 Points

$$f'' = \frac{(x^3+1)^{\cancel{2}}\cdot 12x - 6x^2\cdot 2\cancel{(x^3+1)}\cdot 3x^2}{(x^3+1)^{\cancel{4}\,3}}$$

$$= \frac{12x^4 + 12x - 36x^4}{(x^3+1)^3}$$

4

$$= \frac{-24x^4 + 12x}{(x^3+1)^3} = \frac{12x(-2x^3+1)}{(x^3+1)^3}$$

2

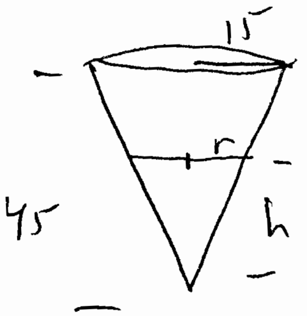
X	X	x^3+1	$-2x^3+1$	f''	2	2
$(-\infty,-1)$	−	−	+	+	up	
$(-1,0)$	−	+	+	−	down	inflection at $x=0$, $1/2^{1/3}$
$(0, 1/2^{1/3})$	+	+	+	+	up	
$(1/2^{1/3}, \infty)$	+	+	−	−	down	

-1 0 $1/2^{1/3}$

25 Points

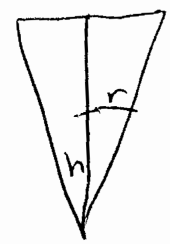
15. A water tank is in the shape of an inverted cone with radius 15 feet and depth 45 feet. If water is pouring into the tank at the rate of 10 cubic feet per minute, at what rate is the depth of the water changing when the water is 25 feet deep?

10 Points



$\frac{dv}{dt} = 10 \ ft^3/min$ 1

Find $\frac{dh}{dt}$ @ $h = 25$ 1



$\frac{r}{h} = \frac{15}{45} = \frac{1}{3} \Rightarrow r = \frac{h}{3}$ 1

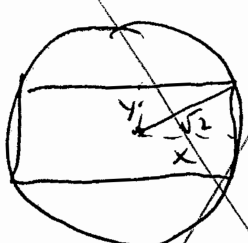
$V = \frac{1}{3}\pi r^2 h$ 1

$= \frac{1}{3}\pi \cdot \frac{h^3}{9} = \frac{1}{27}\pi h^3$ 2

$\frac{dv}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt}$ 3 $\Rightarrow \frac{dh}{dt} = \frac{10 \frac{ft^3}{min} \cdot 9}{\pi \cdot (25^2) ft^2}$ 1

16. Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius $\sqrt{2}$.

10 Points



$A = 4xy$ 2

$x^2 + y^2 = 2$ 2

$\Rightarrow y = \sqrt{2 - x^2}$

$A = 4x \cdot \sqrt{2 - x^2}$ 2

$A' = 4\sqrt{2-x^2} + 4x \cdot \frac{1}{2}(2 - x^2)^{-1/2} \cdot (-2x) = 0$

$\Rightarrow 4(2 - x^2) + (-4x^2) = 0$ 2

$\Rightarrow 8 - 8x^2 = 0$ 1

$\Rightarrow x = 1, \ y = 1$

X	A'	
(0, 1)	+	inc
(1, $\sqrt{2}$)	−	dec

max 1

20 Points