150 Points

Calc I - 3450:221
FINAL Fall 98

NAME Key
ROW

Show **ALL** your work.

1. Find the average value of the function $f(x) = x^2 - 2x + 2$ over $x \in [-2, 1]$.

8 Points

$$\frac{1}{1-(-2)} \int_{-2}^{1} (x^2 - 2x + 2)\, dx$$

1 1 1

$$= \frac{1}{3}\left[\frac{x^3}{3} - x^2 + 2x\right]_{-2}^{1}$$

3

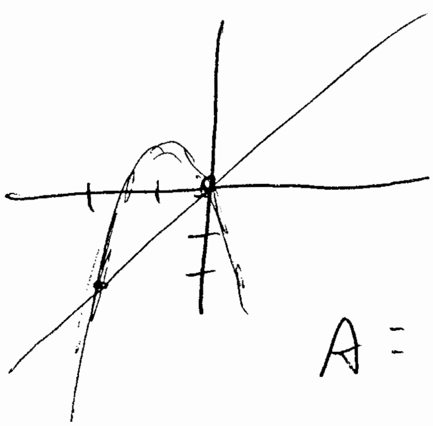
$$= \frac{1}{3}\left[\left(\frac{1}{3} - 1 + 2\right) - \left(\frac{-8}{3} - 4 - 4\right)\right] = \frac{1}{3}\left[3 + 1 + 8\right] = 4$$

2

2. SET UP integral(s) to find the area of the region bounded by $y = x$ and $y = -x - x^2$.

6 Points

$$x = -x - x^2 \Rightarrow x^2 + 2x = 0 \Rightarrow x = 0, -2$$



$$A = \int_{-2}^{0} \left[(-x - x^2) - x\right] dx$$

2 2 2

14 Points

6. Consider the region bounded by the graphs of $y = x - 1$ and $x = (y-1)^2$ for $y \geq 1$.

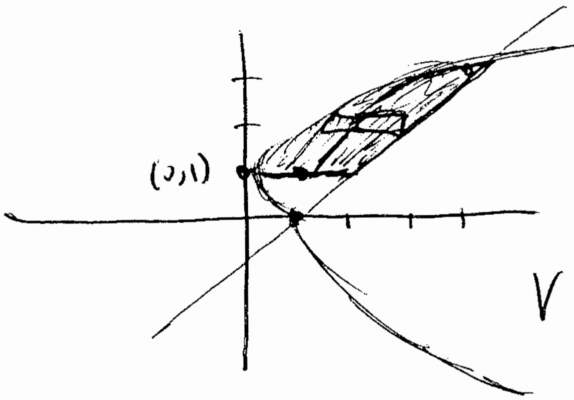
SET UP THE INTEGRAL(S) needed to find the volume of the solid of revolution formed by revolving this region about the

a) x-axis (Set up the integral(s) for integration with respect to y.)

7 Points

So $x = y+1 \Rightarrow y+1 = (y-1)^2 = y^2 - 2y + 1$

$y^2 - 3y = 0 \Rightarrow y = 0, 3$



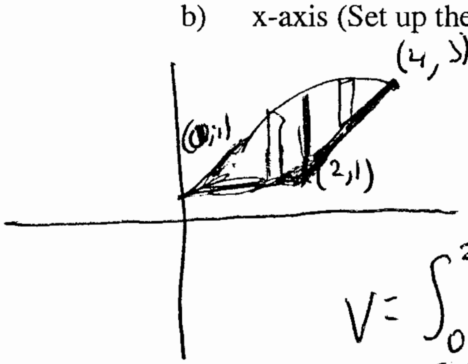
shells

$$V = \int_1^3 2\pi\, y \cdot \left[(y+1) - (y-1)^2\right] dy$$

2 1 1 3

b) x-axis (Set up the integral(s) for integration with respect to x.)

8 Points



washers

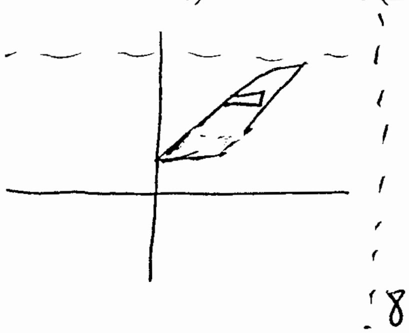
$\sqrt{x} = y - 1 \Rightarrow y = 1 + \sqrt{x}$

$$V = \int_0^2 \pi\left[(1+\sqrt{x})^2 - 1\right] dx + \int_2^4 \pi\left[(1+\sqrt{x})^2 - (x-1)^2\right] dx$$

2 2 2 2

c) line x = 8 (Indicate the method used.)

7 Points



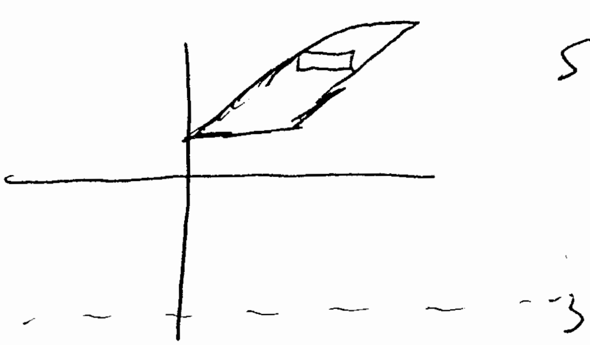
washer

$$V = \int_1^3 \pi\left[\left(8 - (y-1)^2\right)^2 - \left(8 - (y+1)\right)^2\right] dy$$

2 1 2 2

d) line y = -3 (Indicate the method used.)

8 Points
30 Points



Shell

$$V = \int_1^3 2\pi (y+3)\left[(y+1) - (y-1)^2\right] dy$$

2 1 2 3

7. Evaluate $\int_{1}^{4} \frac{x^3 - 2x + 3}{x\sqrt{x}}\, dx.$ $\quad \frac{x^3 - 2x + 3}{x^{3/2}}$

7 Points

$= \int_1^4 \left[x^{3/2} - 2x^{-1/2} + 3x^{-3/2}\right] dx$ 2

$= \left[\frac{2}{5}x^{5/2} - 4x^{\frac{1}{2}} - 6x^{-1/2}\right]_1^4$ 3

$= \left(\frac{2}{5}\cdot 32 - 4\cdot 2 - \frac{6}{2}\right) - \left(\frac{2}{5} - 4 - 6\right)$

$= \frac{62}{5} - 8 - 3 + 4 + 6 = \frac{62}{5} - 1 = 57/5$ 2

8. Evaluate $\int_{-4}^{9} x^4(7 - 3x^5)^6\, dx + \int_{9}^{-4} x^4(7 - 3x^5)^6\, dx.$

7 Points

0

9. Evaluate $\int \frac{4x + 10\cos(5x)}{[x^2 + \sin(5x)]^3}\, dx.$

8 Points

let $u = x^2 + \sin(5x)$ 8

$du = [2x + 5\cos 5x]\,dx \Rightarrow 2du = [4x + 10\cos 5x]\,dx$

$\int \frac{2du}{u^3} = -u^{-2} + C$

3 2

$= -\left[x^2 + \sin 5x\right]^{-2} + C$

22 Points

10. Evaluate $\int [\, x^3 + 7x \sec^2(2x^2)\,]dx$.

8 Points

$$= \int x^3\, dx + \int 7x \sec^2(2x^2)\, dx$$

3

let $u = 2x^2$

$du = 4x\, dx \Rightarrow \frac{du}{4} = x\,dx$

$$= \frac{x^4}{4} + \int \frac{7}{4} \sec^2 u\, du$$

2

$$= \frac{x^4}{4} + \frac{7}{4} \tan u + C = \frac{x^4}{4} + \frac{7}{4} \tan(2x^2) + C$$

1

2

11. Evaluate $\int_{-2}^{-1} x^2\sqrt{x+2}\ dx$.

8 Points

let $u = x + 2$ 3

$du = dx$

$x = -2 \Rightarrow u = 0$

$x = -1 \Rightarrow u = 1$

$x = u - 2$

3

$$\int_0^1 (u-2)^2 \sqrt{u}\, du$$

$$= \int_0^1 (u^2 - 4u + 4) u^{1/2}\, du = \int_0^1 [u^{5/2} - 4u^{3/2} + 4u^{1/2}]\, du$$

2

$$= \frac{2}{7} u^{7/2} - \frac{8}{5} u^{5/2} + \frac{8}{3} u^{3/2} \Big|_0^1$$

$$= 2/7 - \frac{8}{5} + \frac{8}{3} = \frac{30 - 168 + 280}{105} = \frac{142}{105}$$

1

16 Points

12. Evaluate $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\left(\frac{i}{n}\right)^{7}\frac{1}{n}$. $= \frac{b-a}{n}$

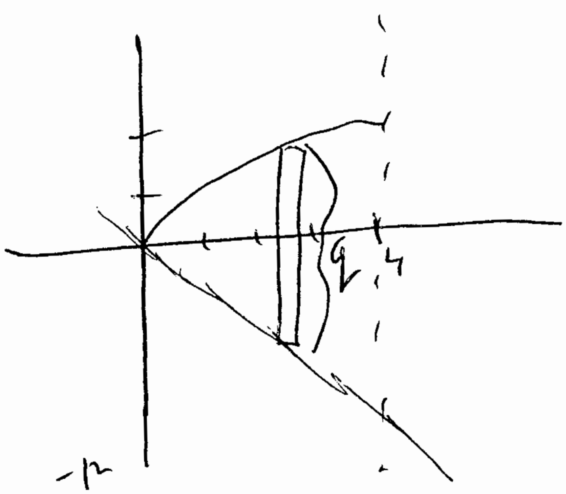
$\left(0+\frac{1}{n}\cdot i\right)^{7}$

$= \int_0^1 x^7\,dx = \frac{x^8}{8}\Big|_0^1 = \frac{1}{8}$

5 2

7 Points

13. A solid has as its base the region in the plane bounded by $y = \sqrt{x}$, $x = 4$ and $y = -3x$. Each cross section perpendicular to the plane and parallel to the y–axis is a rectangle with height five times the length of the base which lies in the plane. SET UP integral(s) to find the volume of this solid.



$A = 5q^2$

7 Points

$V = \int_0^4 5\cdot\left[\sqrt{x}-(-3x)\right]^2 dx$

2 1 4

14 Points

14. Answer all of the questions below for $f(x) = \dfrac{x^2+1}{x^2-1}$.

a) Find any vertical, horizontal, and slant asymptotes. Show all limits used.

6 Points

$$\left.\begin{aligned}\lim_{x\to\infty} f(x) &= 1\\ \lim_{x\to-\infty} f(x) &= 1\end{aligned}\right\} y=1 \text{ HA}$$

2

$\lim_{x\to 1^+} f(x) = \infty$ $\lim_{x\to -1^+} f(x) = -\infty$ VA at $x = \pm 1$

$\lim_{x\to 1^-} f(x) = -\infty$ $\lim_{x\to -1^-} f(x) = \infty$

4

b) Find the intervals where f(x) is increasing/decreasing and identify any extrema.

6 Points

$$f' = \frac{(x^2-1)\cdot 2x - (x^2+1)\cdot 2x}{(x^2-1)^2}$$

$$= \frac{-4x}{(x^2-1)^2}$$

2

x	$-4x$	f'	
$(-\infty,-1)$	+	+	inc
$(-1,0)$	+	+	inc
$(0,1)$	−	−	dec
$(1,\infty)$	−	−	dec

rel max at $x=0$

4

c) Find the intervals where f(x) is concave up/down and identify any points of inflection.

6 Points

$$f'' = \frac{(x^2-1)^2\cdot(-4) + 4x(2)(x^2-1)2x}{(x^2-1)^4}$$

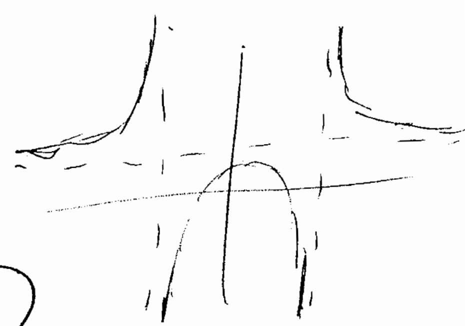
$$= \frac{-4(x^2-1)+16x^2}{(x^2-1)^3}$$

$$= \frac{12x^2+4}{(x^2-1)^3}$$

2

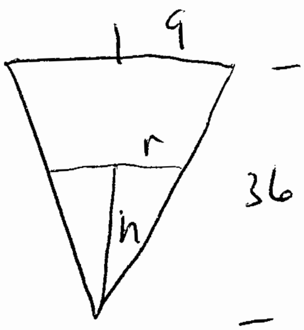
X	$(x^2-1)^3$	f''
$(-\infty,-1)$	+	up
$(-1,1)$	−	down
$(1,\infty)$	+	up

4



18 Points

15. A water tank is in the shape of an inverted cone with radius 9 feet and depth 36 feet. If water is pouring into the tank at the rate of 5 cubic feet per minute, at what rate is the depth of the water changing when the water is 10 feet deep?



$$\frac{dv}{dt} = 5 \frac{ft^3}{min}$$

7 Points

$$V = \frac{1}{3}\pi r^2 h \quad 2$$

$$\frac{r}{h} = \frac{9}{36} = \frac{1}{4}$$

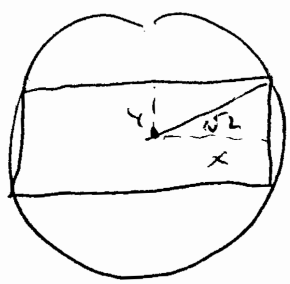
$$\Rightarrow r = \frac{1}{4}h \quad 2$$

$$\text{So } V = \frac{1}{3}\pi \cdot \frac{1}{16} h^3 = \frac{1}{48}\pi h^3$$

$$\frac{dv}{dt} = \frac{1}{16}\pi h^2 \frac{dh}{dt} \quad 2$$

$$\Rightarrow \frac{dh}{dt} = \frac{dv}{dt} \cdot \frac{16}{\pi h^2} = \frac{5\frac{ft^3}{min} \cdot 16}{\pi \cdot (10\,ft)^2} \quad 4 \quad = 0.2546$$

16. Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius $\sqrt{2}$.



$$A = 4xy \quad 2 \quad , \quad x^2 + y^2 = 2$$

$$\Rightarrow y = \sqrt{2 - x^2} \quad 2$$

7 Points

$$A = 4x \cdot \sqrt{2 - x^2}$$

$$A' = 4\sqrt{2 - x^2} + 4x \cdot \tfrac{1}{2}(2 - x^2)^{-1/2} \cdot (-2x) = 0$$

$$\Rightarrow 4(2 - x^2) - 4x^2 = 0 \quad 2$$

$$\Rightarrow 8 - 8x^2 = 0$$

$$\Rightarrow x = \pm 1$$

8 Keep $x = 1$ so $y = \sqrt{2 - 1} = 1$

14 Points