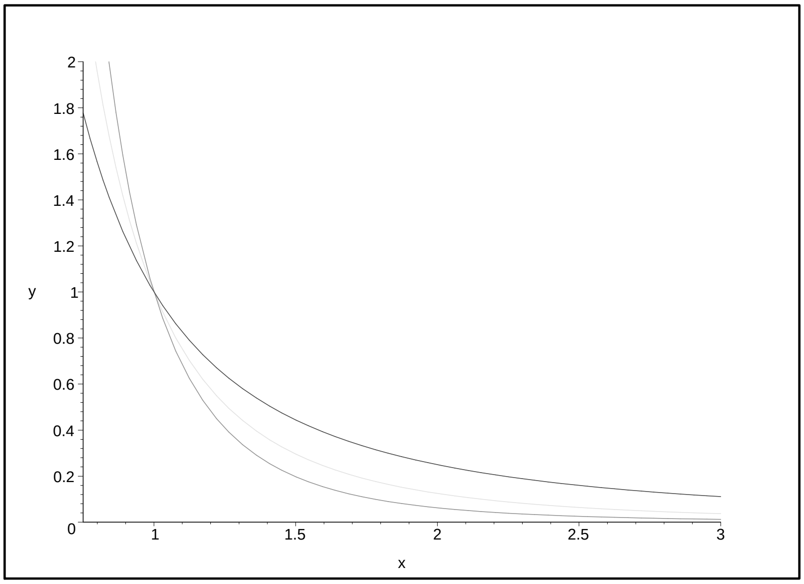
(1) (4 points each)



Above is the graph of three curves. The curve which is on top for $x \geq 1$ is the graph of $y = f(x)$ and the bottom curve is the graph of $y = g(x)$. The middle curve is $y = 1/x^2$. The left-most endpoint on the graph is $x = .75$, but all three functions are defined for $x > 0$. We also know that $f(x) \leq 1/x^2 \leq g(x)$ for $0 < x < 1$ and $g(x) \leq 1/x^2 \leq f(x)$ for $1 \leq x$.

Which of the following must be true, might be true or cannot be true? Completely justify your reasoning!

(a) $\int_1^\infty f(x)dx$ converges.
Since $f(x) \geq 1/x^2$ on $(1, \infty)$ and $\int_1^\infty 1/x^2 dx$ converges, we can't tell if $\int_1^\infty f(x)dx$ converges or diverges. So this might be true.

(b) $\int_0^1 g(x)dx$ converges.
$g(x) \geq 1/x^2$ on $(0, 1)$ and $\int_0^1 1/x^2$ diverges. Since $\int_0^1 g(x)dx$ is greater than a divergent integral, it also diverges. Hence, this cannot be true. Note: Since $1/x^2$ has an infinite discontinuity at $x = 0$ and since $g(x) \geq 1/x^2$ on $(0, 1)$, $g(x)$ also has an infinite discontinuity at $x = 0$ so the integral in this problem is improper.

(c) $\int_1^\infty g(x)dx$ converges.
Since $g(x) \leq 1/x^2$ and $\int_1^\infty 1/x^2 dx$ converges, $\int_1^\infty g(x)dx$ converges as well. Hence this must be true.

(2) Let R denote the finite region bounded by the graphs of $f(x) = x$ and $g(x) = x(4 - x)$.

(a) (3 points) Sketch the region R.

The graph of $y = g(x)$ is *above* the graph of $y = f(x)$ when $0 \le x \le 3$.

(b) (5 points) A solid is generated by rotating the region R about the x-axis. Set up, **but do not evaluate**, a definite integral which represents the volume of this solid.

Use the washer method. The radius of the outside of the washer at x is $x(4-x)$ while the radius of the inside of the washer at x is x. Therefore, the integral giving the volume is

$$\int_0^3 (\pi(x(4-x))^2 - \pi x^2)dx.$$

(c) (5 points) A solid is generated by rotating the region R about the line $y = -1$. Set up, **but do not evaluate**, a definite integral (or integrals) which represents the volume of this solid.

Again, use the washer method. The outside radius is $x(4-x) - (-1)$ while the inside radius is $x - (-1)$. Therefore, the volume is given by

$$\int_0^3 (\pi(x(4-x)+1)^2 - \pi(x+1)^2)dx.$$

(d) (2 points) Which of the two solids from parts (b) and (c) would you expect to have larger volume and why?

We expect the solid from part (c) is larger since the region travels a larger distance around the axis $y = -1$ then around $y = 0$.

(3) (a) (5 points) Graph the curve $r = 1 + \sin\theta$ (in polar coordinates) with $0 \le \theta \le 2\pi$.

As θ goes from 0 to $\pi/2$, r goes from 1 to 2. As θ goes from $\pi/2$ to π, r goes from 2 to 1. As θ goes from π to $3\pi/2$, r goes from 1 to 0. And as θ goes from $3\pi/2$ to 2π, r goes from 0 to 1. This gives a cardioid with "center" at the origin.

(b) (6 points) Compute the area bounded by the graph from part (a).

The area is given by $\int_0^{2\pi} \frac{1}{2}(1+\sin\theta)^2 d\theta$. Expanding the integrand, we get

$$\frac{1}{2}\int_0^{2\pi} (1 + 2\sin\theta + \sin^2\theta)d\theta.$$

The integral of the first term is π, the integral of the second term is 0, while, to integrate the third term, we need to compute

$$\int \sin^2\theta d\theta.$$

To compute this, we use integration by parts with $u = \sin\theta$ and $dv = \sin\theta d\theta$. We get

$$\int \sin^2\theta d\theta = -\sin\theta\cos\theta + \int \cos^2\theta d\theta = -\sin\theta\cos\theta + \int (1-\sin^2\theta)d\theta.$$

Using this to isolate $\int \sin^2\theta d\theta$ we find

$$\int \sin^2\theta d\theta = \frac{1}{2}(\theta - \sin\theta\cos\theta) + C.$$

It follows that the last term in the area integral is $\pi/2$ for a total area of $3\pi/2$.

(4) Let R be the region in the first quadrant bounded by $y = \sin x$, $y = \cos x$ and $x = 0$ where x and y are in cm.

(a) (5 points) Assuming the density of R is $7gms/cm^2$, compute the mass of the region.

The mass equals the area times the density, which is the constant 7. The area is $\int_0^{\pi/4}(\cos x - \sin x)dx = \sqrt{2} - 1$. Therefore, the mass is $7(\sqrt{2} - 1)$ grams.

(b) (5 points) Set up **but do not evaluate** $\overline{x}$ if the density of R is $7gms/cm^2$. Include correct units.

The x coordinate of the center of mass is given by

$$\frac{\int_0^{\pi/4} x7l_x(x)dx}{\text{mass}} = \frac{\int_0^{\pi/4} x7(\cos x - \sin x)dx}{7(\sqrt{2}-1)}.$$

(c) (3 points) If the density $\delta(x) = \frac{1}{1+x^2}\ gms/cm^2$, in what direction, if any, would $\overline{x}$ shift from part (a)? Explain without calculating $\overline{x}$.

The x-coordinate of the center of mass should shift to the left since the density decreases as x goes to the right, as opposed to staying constant as x goes to the right.