Math 125 Exam #2 Solutions

(1) (6 points each) Compute the following integrals

(a) $\int \frac{dx}{(9+x^2)^{3/2}}$. This is a trig substitution problem: let $x = 3\tan\theta$. Then $dx = 3\sec^2\theta d\theta$ and so the integral equals

$$\int \frac{3\sec^2\theta d\theta}{(9\sec^2\theta)^{3/2}} = \frac{3}{27}\int \frac{d\theta}{\sec\theta} = \frac{1}{9}\int \cos\theta d\theta = \frac{1}{9}\sin\theta + C.$$

Thus, the original integral is $\frac{1}{9}\sin(\arctan(\frac{x}{9}))+C$. Since $\tan(\arctan\frac{x}{3}) = \frac{x}{3}$, we conclude that $\sin(\arctan(\frac{x}{9})) = \frac{x}{\sqrt{9+x^2}}$. We conclude that the final answer is $\frac{1}{9}\frac{x}{\sqrt{9+x^2}}+C$.

(b) $\int \frac{3x}{(x^2-9)^{3/2}}dx$. The easiest way to do this integral is by substitution: $u = x^2 - 9$ so that $du = 2xdx$. Then the integral is $\frac{3}{2}\int \frac{du}{u^{3/2}} = -3u^{-1/2} + C$. Putting x back in, we find the final answer to be $\frac{-3}{\sqrt{x^2-9}} + C$.

(c) $\int \frac{3x+2}{(x+4)(x+2)}dx$. This is solved using partial fraction decomposition. One must solve for A and B in the equation:

$$\frac{3x+2}{(x+4)(x+2)} = \frac{A}{x+4} + \frac{B}{x+2}$$

which implies that $3x + 2 = A(x + 2) + B(x + 4)$. Therefore, $3 = A + B$ and $2 = 2A + 4B$. It follows that $A = 5$ and $B = -2$ so that the original integral is $5\ln|x+4| - 2\ln|x+2| + C$.

(d) $\int x\sin(2x)dx$. To solve this, use integration by parts with $u = x$ and $dv = \sin(2x)dx$. Then $du = dx$ and $v = \frac{-1}{2}\cos(2x)$. The answer is $\frac{-x}{2}\cos(2x) + \frac{1}{4}\sin(2x) + C$.

(2) The integral $\int_2^3 \frac{dx}{(x-2)^{3/2}}$ is improper since the integrand has an infinite discontinuity at $x = 2$. Therefore, it equals

$$\lim_{k\to 2^+}\int_k^3 \frac{dx}{(x-2)^{3/2}}.$$

Letting $u = x - 2$ we get that the integral $\int \frac{dx}{(x-2)^{3/2}}$ equals $\int \frac{du}{u^{3/2}} = -2\frac{1}{\sqrt{u}} + C$. Therefore, the original integral equals

$$\lim_{k\to 2^+}(-2 - (-\frac{2}{\sqrt{k-2}})).$$

Since the denominator of the second term goes to 0, the second term goes to ∞ so that the original integral diverges.

(3) The integral $\int \frac{dx}{(4-9x^2)^{3/2}}$ is almost a trig substitution integral, except there is a $9x^2$ instead of just an x^2. To put it in the correct form substitute $u = 3x$ so that $du = 3dx$. The integral becomes $\frac{1}{3}\int \frac{du}{(4-u^2)^{3/2}}$. Now we can use $u = 2\sin\theta$ and our u-integral becomes

$$\frac{1}{3}\int \frac{2\cos\theta}{8\cos^3\theta}d\theta = \frac{1}{12}\int \sec^2\theta d\theta = \frac{1}{12}\tan\theta + C = \frac{1}{12}\tan(\arcsin\frac{u}{2}) + C.$$

This last expression is $\frac{1}{12}\frac{u}{\sqrt{4-u^2}} + C$. Putting x back in we get

$$\frac{1}{12}\frac{3x}{\sqrt{4-9x^2}} + C.$$

(4) (8 points) Consider the region bounded by the curves $y = x + 1/x^2$ and $y = x - 1/x^2$ for $x \geq 1$.

The area equals

$$\int_1^\infty (x + \frac{1}{x^2}) - (x - \frac{1}{x^2}))dx = \int_1^\infty \frac{2}{x^2}dx.$$

This integral equals

$$\lim_{l\to\infty}\int_1^l \frac{2}{x^2}dx = 2\lim_{l\to\infty}(\frac{-1}{x}|_1^l) = 2\lim_{l\to\infty}(\frac{-1}{l} + 1) = 2.$$

We conclude that the area is finite and equal to 2.