Math 125 Exam #1 Solutions

(1) (5 points each) Compute the following:

(a)

(b) $\int_0^2 (x^3 - 2x^2 + 1)dx$.

This equals $\int_0^2 x^3 dx - 2\int_0^2 x^2 + \int_0^2 dx = \frac{x^4}{4}|_0^2 - 2\frac{x^3}{3}|_0^2 + x|_0^2 = 4 - \frac{16}{3} + 2$.

(c) $\int_1^2 \frac{1+y^2}{y} dy$.

No u-substitution is required. We just simplify the integrand:

$$\int_1^2 \frac{1+y^2}{y} dy = \int_1^2 (y^{-1} + y)dy = (\ln|y| + y^2/2)|_1^2 = \ln 2 + 2 - (\ln 1 + 1/2) = \ln 2 + 3/2.$$

(d) $\frac{d}{dt}\int_t^{t^2} \sin(x^2)dx$.

Using properties of integrals, we know this can be broken up into the sum

$$\frac{d}{dt}\int_t^0 \sin(x^2)dx + \frac{d}{dt}\int_0^{t^2} \sin(x^2)dx.$$

By the second FTC, this equals $-\sin(t^2) + \sin((t^2)^2) \cdot 2t$.

(2) (5 points each) The following questions concern the area under the curve $f(x) = 1 + x^2$ and above the x-axis between $x = -1$ and $x = 2$.

(a) Use a left-handed sum with 3 rectangles to estimate the area. Draw a picture that illustrates this estimate.

There are three rectangles, each of length $(b - a/3) = 2 - (-1)/3 = 1$. The area of the rectangles is

$1 \cdot f(-1) + 1 \cdot f(0) + 1 \cdot f(1) = 5$.

(b) Use a right-handed sum with 3 rectangles to estimate the area. Draw a picture that illustrates this estimate.

As above, the area is

$1 \cdot f(0) + 1 \cdot f(1) + 1 \cdot f(2) = 8$.

(c) Calculate the exact area.

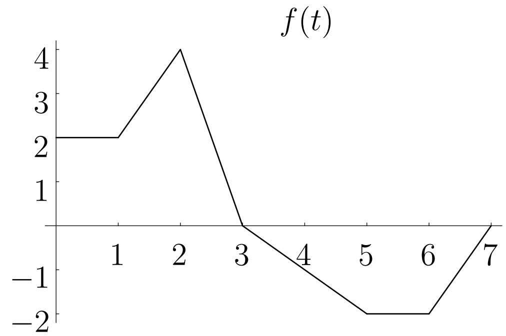
$\int_{-1}^2 (1 + x^2)dx = (x + x^3/3)|_{-1}^2 = 2 + 8/3 - (-1 - 1/3) = 6$.

(3) (a) Compute the area enclosed above by $y = 4$ and below by $y = x^2$.

The curves intersect at x-values in which $4 = x^2$, i.e. at $x = 2, -2$. Therefore the area is given by $\int_{-2}^2 4 - x^2 dx = (4x - x^3/3)|_{-2}^2 = 32/3$.

(b) Find the horizontal line $y = k$ that divides the area between $y = 4$ and $y = x^2$ found in part (a) into two equal parts.

One half the area above is $16/3$, and the area between the line $y = k$ and the parabola $y = x^2$ is given by the definite integral $\int_?^? k - x^2 dx$. How do we find the limits of integration? As above, the limits correspond to the x-values of the



intersection points of the two curves $y = k$ and $y = x^2$. These curves intersect where $k = x^2$, i.e. when $x = \sqrt{k}, -\sqrt{k}$. Thus, we must solve for k in the integral

$$16/3 = \int_{-\sqrt{k}}^{\sqrt{k}} k - x^2 dx.$$

The integral equals $(kx - x^3/3)|_{-\sqrt{k}}^{\sqrt{k}} = 4/3k^{3/2}$. On the other hand, this must equal $16/3$ so we have the equation

$$16/3 = 4/3k^{3/2}$$

which implies that $k = 2^3\sqrt{2}$.

(4) (5 points each) Let $g(x) = \displaystyle\int_0^x f(t)\, dt$, where $f(t)$ is the function whose graph is shown above.

(a) Evaluate $g(0)$, $g(3)$, and $g'(3)$.
$g(0) = \int_0^0 f(t)dt = 0$, $g(3) = \int_0^3 f(t)dt$. To get this integral, do not use the fundamental theorem of calculus, use the fact that the definite integral gives the signed area corresponding to the graph. Hence, $g(3) = 7$ (the area above the x-axis to the left of $x = 3$.
Finally, $g'(x) = d/dt \int_0^x f(t)dt = f(x)$ so $g'(3) = f(3) = 0$.

(b) On what interval is g increasing? Why? g is increasing when $g'(x) = f(x) > 0$. Looking at the graph, this occurs in the interval $(0, 3)$.

(c) Where does g have its maximum value in the interval $0 \le t \le 7$? Why?
This occurs at $x = 3$, since $g(3)$ is the area to the left of $x = 3$ and after this point, one must subtract areas.