Solutions to MA242 Quiz 9, 11/14/06

1. Let the subspace H of $\mathbb{R}^4$ be defined by

$$H := \left\{ \begin{pmatrix} a+b \\ 2a \\ 3a-b \\ -b \end{pmatrix} : \ a, b \text{ in } \mathbb{R} \right\}.$$

(a) Find a basis for H.
(b) State the dimension of H. (*Justify* your answer!)

Solution: (a) Since for $\mathbf{x}$ in H,

$$\mathbf{x} = a \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \end{pmatrix} = a\mathbf{v}_1 + b\mathbf{v}_2,$$

H is the span of $\{\mathbf{v}_1, \mathbf{v}_2\}$. Since neither $\mathbf{v}_1$ nor $\mathbf{v}_2$ are zero and since they are not multiples of each other, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, and is therefore a basis for H.

(b) Since the basis for H found in (a) consists of two vectors, the dimension of H is 2.

2. Assume that the matrix A is row equivalent to B, where

$$A = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Determine Rank$A$ and dim Nul$A$. (*Justify* your answer!)
(b) Find bases for Row$A$ and Nul$A$.

Solution: (a) Since the matrix B is in echelon form, we see that there are 3 pivot columns: 1,3, and 5; hence, the dimension of Col$A$ (the rank) is 3. Since there are two columns without pivots, the system $A\mathbf{x} = \mathbf{0}$ has two free variables; hence, the dimension of Nul$A$ is 2.

(b) A basis for Row$A$ is obtained from the pivot rows of B, $(1, -3, 0, 5, -7)$, $(0, 0, 2, -3, 8)$, and $(0, 0, 0, 0, 5)$. To find a basis for Nul$A$, reduce B to reduced echelon form:

$$B \sim \begin{bmatrix} 1 & -3 & 0 & 5 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the general solution of $A\mathbf{x} = \mathbf{0}$ is $x_1 = 3x_2 - 5x_4$, $x_3 = \frac{3}{2}x_4$, and $x_5 = 0$, with x_2 and x_4 free. A basis for Nul$A$ is given by $(3, 1, 0, 0, 0)$ and $(-5, 0, \frac{3}{2}, 1, 0)$.