Solutions to MA242 Quiz 8, 11/07/06

1. Let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 6 \\ 2 \\ -1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_4 = \begin{pmatrix} -4 \\ -8 \\ 9 \end{pmatrix}.$$

(a) Explain why H is a subspace of $\mathbb{R}^3$. (*Justify* your answer!)
(b) Find a basis for H.
(c) Using your answer in (b), explain why H is isomorphic to $\mathbb{R}^2$. (*Justify* your answer!)

Solution: (a) Since the span of a set of vectors in a vector space V always is a subspace of V (Theorem 1, Section 4.1) and since H is defined as $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, H is a subspace of $\mathbb{R}^3$.

(b) Write the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_4$ into a matrix A and then row reduce A to find its pivot columns:

$$A = [\mathbf{v}_1 \ \ldots \mathbf{v}_4] = \begin{bmatrix} 1 & 6 & 2 & -4 \\ -3 & 2 & -2 & -8 \\ 4 & -1 & 3 & 9 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 6 & 2 & -4 \\ 0 & 5 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first two columns of A are the pivot columns and hence form a basis of Col$A = H$; in other words, $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for H.

(c) Since any vector $\mathbf{x}$ in H can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, that is, since

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$

for some scalars c_1, c_2, it follows that the coordinate vector $[\mathbf{x}]_{\mathcal{B}} = (c_1, c_2)$ of $\mathbf{x}$ is in $\mathbb{R}^2$. Hence, H is isomorphic to $\mathbb{R}^2$ under the coordinate mapping, see Theorem 8 in Section 4.4.

2. Let

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} -3 \\ 4 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{b}_3 = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix}.$$

(a) Show that the set $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for $\mathbb{R}^3$.
(b) Find the change-of-coordinate matrix $P_{\mathcal{B}}$ from $\mathcal{B}$ to the standard basis $\mathcal{E}$.
(c) Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of $\mathbf{x} = (-8, 2, 3)$ relative to $\mathcal{B}$.

Solution: (a) The change-of-coordinates matrix $P_{\mathcal{B}} = [\mathbf{b}_1 \ \mathbf{b}_2 \ \mathbf{b}_3]$ is row-equivalent to the identity matrix I_3. Hence, by the Invertible Matrix Theorem, $P_{\mathcal{B}}$ is invertible, and its columns form a basis for $\mathbb{R}^3$.

(b) By (a), it follows that

$$P_{\mathcal{B}} = \begin{bmatrix} 1 & -3 & 3 \\ 0 & 4 & -6 \\ 0 & 0 & 3 \end{bmatrix}$$

for the change-of-coordinates matrix $P_{\mathcal{B}}$, with $[\mathbf{x}]_{\mathcal{E}} = \mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$.

(c) To solve the equation $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}$ for $[\mathbf{x}]_{\mathcal{B}}$, row reduce the augmented matrix $[P_{\mathcal{B}}\ \mathbf{x}]$:

$$\begin{bmatrix} 1 & -3 & 3 & -8 \\ 0 & 4 & -6 & 2 \\ 0 & 0 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Hence, $[\mathbf{x}]_{\mathcal{B}} = (-5, 2, 1)$.