Solutions to MA242 Quiz 7, 10/31/06

1. Let S be the parallelogram determined by $\mathbf{b}_1 = (4, -7)$ and $\mathbf{b}_2 = (0, 1)$, and let
$$A = \begin{bmatrix} 7 & 2 \\ 1 & 1 \end{bmatrix}.$$
Compute the area of the image of S under the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

Solution: Since the area of S is given by $|\mathrm{det}B| = |\mathrm{det}[\mathbf{b}_1,\ \mathbf{b}_2]|$ and since the area of the image of S under $\mathbf{x} \mapsto A\mathbf{x}$ is given by
$$(\text{area of image of } S \text{ under } \mathbf{x} \mapsto A\mathbf{x}) = |\mathrm{det}A| \cdot (\text{area of } S),$$
it follows that
$$(\text{area of image of } S \text{ under } \mathbf{x} \mapsto A\mathbf{x}) = |7(1) - 1(2)| \cdot |4(1) - (-7)0| = 5(4) = 20.$$

Note: Another possibility to compute that area is to first find the image of S under $\mathbf{x} \mapsto A\mathbf{x}$ by computing AB, and then to find $|\mathrm{det}(AB)| = 20$.

2. Let
$$A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}.$$
(a) Determine if $\mathbf{w}$ is in Col$A$.
(b) Is $\mathbf{w}$ in Nul$A$?

Solution: (a) To find out whether $\mathbf{w}$ is in Col$A$, we have to check whether $\mathbf{w}$ can be written as a linear combination of the columns of A. In other words, we have to determine whether $[A\ \mathbf{w}]$ is consistent:
$$[A\ \mathbf{w}] \sim \begin{bmatrix} -8 & -2 & -9 & 2 \\ 6 & 4 & 8 & 1 \\ 4 & 0 & 4 & -2 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 1 & -\frac{1}{2} \\ 0 & -2 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$
Hence, the linear system is consistent, which means that $\mathbf{w}$ *is* in Col$A$.

Note: It is *not* correct to claim that $\mathbf{w}$ is not in Col$A$ because it is not a column of A (or a multiple of a column of A); Col$A$ is the space *spanned* by the columns of A.

(b) To determine whether $\mathbf{w}$ is in Nul$A$, we *only* have to check whether it is a solution of $A\mathbf{x} = \mathbf{0}$:
$$A\mathbf{w} = \begin{pmatrix} -8(2) - 2(1) - 9(-2) \\ 6(2) + 4(1) + 8(-2) \\ 4(2) + 0(1) + 4(-2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, $\mathbf{w}$ also is in Nul$A$.

Note: It is *not* necessary to *solve* the linear system $A\mathbf{x} = \mathbf{0}$, since the question was not to find a spanning set for Nul$A$, but only to check whether one specific vector is in that space.

3. Is the set of all polynomials of the form $\mathbf{p}(t) = a + t^2$ (with a in $\mathbb{R}$) a subspace of $\mathbb{P}_2$? (*Justify* your answer!)

   Solution: The answer is no; the simplest justification is because the zero vector is not contained in the set: $a + t^2$ is *not* the zero polynomial for any choice of a. (Alternatively, one can argue that the set is not closed under vector addition and scalar multiplication.)