Solutions to MA242 Quiz 6, 10/24/06

1. (a) Compute the determinant $\det A$ of the matrix

$$A = \begin{bmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -6 & -7 & 5 \\ 5 & 0 & 4 & 4 \end{bmatrix}.$$

(*Show* all work!)
(b) Using the result of (a), determine whether the columns of A are linearly independent. (*Justify* your answer!)

Solution: (a) Doing a cofactor expansion along the second row, we find

$$\det A = (-1)^{2+3} 3 \begin{vmatrix} 1 & -2 & 2 \\ 2 & -6 & 5 \\ 5 & 0 & 4 \end{vmatrix};$$

now, one possibility is to compute the 3×3-determinant a cofactor expansion across the third row, say,

$$\det A = -3\left((-1)^{3+1} 5 \begin{vmatrix} -2 & 2 \\ -6 & 5 \end{vmatrix} + (-1)^{3+3} 4 \begin{vmatrix} 1 & -2 \\ 2 & -6 \end{vmatrix}\right) = -3\big(5(2) + 4(-2)\big) = -6.$$

(Other possibilities are to expand down the second column or use the formula for 3×3-matrices from the Problem Section in 3.1.)
(b) Since $\det A \neq 0$, the matrix A is invertible. Hence, by the Invertible Matrix Theorem, the columns of (a) are linearly independent.

2. (a) Find the general solution of $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = (-2, 3)$ by using the given LU factorization for A,

$$A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & -5 \\ 0 & -1 & 2 \end{bmatrix}.$$

(b) Write the solution vector $\mathbf{x}$ from (a) in parametric vector form, and interpret it geometrically.

Solution: (a) Since $A\mathbf{x} = (LU)\mathbf{x} = L(U\mathbf{x}) = \mathbf{b}$, we first write $\mathbf{y} = U\mathbf{x}$ and solve $L\mathbf{y} = \mathbf{b}$:

$$[L\ \mathbf{b}] = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix}.$$

Hence, $\mathbf{y} = (-2, -1)$. In the second step, we solve $U\mathbf{x} = \mathbf{y}$:

$$[U\ \mathbf{y}] = \begin{bmatrix} 1 & 3 & -5 & -2 \\ 0 & -1 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -5 \\ 0 & 1 & -2 & 1 \end{bmatrix}.$$

Hence, $x_1 = -5 - x_3$ and $x_2 = 1 + 2x_3$ are basic variables, and x_3 is free.

(b) In parametric vector form, $\mathbf{x}$ is given by

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -5 - x_3 \\ 1 + 2x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix},$$

which defines a line in $\mathbb{R}^3$ which passes through the point $(-5, 1, 0)$.