Solutions to MA242 Quiz 5, 10/17/06

1. (a) Find the inverse A^{-1} of the matrix

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}.$$

(b) Use the inverse found in (a) to solve the linear system

$$\begin{aligned} x_1 - 2x_3 &= -1 \\ -3x_1 + x_2 + 4x_3 &= 2 \\ 2x_1 - 3x_2 + 4x_3 &= 1. \end{aligned}$$

(c) Is the solution found in (b) unique? (*Justify* your answer!)

Solution: (a) To find the inverse, we apply the algorithm from Section 2.2:

$$[A\ I_3] \sim \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}.$$

Therefore,

$$A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}.$$

(b) To find $\mathbf{x}$ in $A\mathbf{x} = \mathbf{b}$, note that

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}.$$

(c) Yes, $\mathbf{x} = (-1, -1, 0)$ is the unique solution of $A\mathbf{x} = \mathbf{b}$. To justify this, cite either Theorem 5 from Section 2.2, *or* use the fact that by (a), A has a pivot in every column and that therefore $A\mathbf{x} = \mathbf{b}$ has no free variables, *or* apply the Invertible Matrix Theorem to argue that the linear transformation corresponding to $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one and onto.

2. If the equation $A\mathbf{x} = \mathbf{b}$ has more than one solution for some $\mathbf{b}$ in $\mathbb{R}^n$, can the columns of A span $\mathbb{R}^n$? (*Justify* your answer!)

Solution: If $A\mathbf{x} = \mathbf{b}$ has more than one solution, the matrix A cannot be invertible by Theorem 5 in Section 2.2. So, by the Invertible Matrix Theorem, the columns of A cannot span $\mathbb{R}^n$.