Solutions to MA242 Quiz 4, 10/03/06

1. How many rows and columns must a matrix A have in order to define a mapping from $\mathbb{R}^3$ into $\mathbb{R}^4$ by the rule $T(\mathbf{x}) = A\mathbf{x}$? (*Justify* your answer!)

   Solution: A must have 4 rows and 3 columns. For the domain of T to be $\mathbb{R}^3$, A must have 3 columns so that the matrix-vector product $A\mathbf{x}$ is defined for $\mathbf{x}$ in $\mathbb{R}^3$. For the codomain of T to be $\mathbb{R}^4$, the columns of A must have 4 entries (that is, A must have 4 rows), since $A\mathbf{x}$ is a linear combination of the columns of A.

2. Let the transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

   $$T(\mathbf{x}) = T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3, x_2 - 6x_3).$$

   (a) Show that T is a linear transformation by finding its standard matrix.
   (b) Determine if T is one-to-one and onto. (*Justify* your answers!)

   Solution: (a) Since $T(\mathbf{x})$ has two entries, the standard matrix A has 2 rows; since $\mathbf{x}$ has 3 entries, A has 3 columns. To find A, write

   $$\begin{aligned} T(\mathbf{x}) &= \begin{pmatrix} x_1 - 5x_2 + 4x_3 \\ x_2 - 6x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -5 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 4 \\ -6 \end{pmatrix} \\ &= \begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \end{aligned}$$

   Since every matrix transformation is linear and since we have just shown that $T(\mathbf{x}) = A\mathbf{x}$, T is linear.

   (b) The columns of A are linearly dependent, since A has more columns than rows. So, T is *not* one-to-one, see Theorem 12. Moreover, since A has a pivot in each row, the rows of A span $\mathbb{R}^2$. Hence, again by Theorem 12, T maps $\mathbb{R}^3$ onto $\mathbb{R}^2$, that is, it *is* onto.