Solutions to MA242 Quiz 3, 09/26/06

1. Describe the solution sets of $x_1 - 3x_2 + 5x_3 = 0$ and of $x_1 - 3x_2 + 5x_3 = 4$ in parametric vector form. Then, interpret and compare them geometrically.

Solution: Solve the nonhomogeneous equation $x_1 - 3x_2 + 5x_3 = 4$ for the basic variable x_1, so that $x_1 = 4 + 3x_2 - 5x_3$, with x_2 and x_3 free. Writing this solution in vector form, we get

$$\begin{aligned}\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 4 + 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -5x_3 \\ 0 \\ x_3 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}.\end{aligned}$$

Similarly, the solution of the homogeneous equation $x_1 - 3x_2 + 5x_3 = 0$ is

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} := x_2\mathbf{u} + x_3\mathbf{v}.$$

Since the vectors $\mathbf{u}$ and $\mathbf{v}$ are not multiples of each other, the solution set of the homogeneous equation is the *plane* through the origin in $\mathbb{R}^3$ spanned by $\mathbf{u}$ and $\mathbf{v}$. The solution set of the nonhomogeneous equation is also a plane which is parallel to the first one and which passes through the point $\mathbf{p} = (4, 0, 0)$.

2. Let

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -2 \\ 10 \\ 6 \end{pmatrix}, \quad \text{and } \mathbf{v}_3 = \begin{pmatrix} 2 \\ -9 \\ h \end{pmatrix}.$$

(a) For what values of h is $\mathbf{v}_3$ in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$?
(b) For what values of h is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly *independent*?

Solution: (a) The vector $\mathbf{v}_3$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ if and only if the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{v}_3$ has a solution. Hence, row reduce $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$, considered as an augmented matrix, to find

$$\begin{bmatrix} 1 & -2 & 2 \\ -5 & 10 & -9 \\ -3 & 6 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & h+6 \end{bmatrix}.$$

The equation $0 = 1$ shows that the original vector equation has no solution. Therefore, $\mathbf{v}_3$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ for *no* value of h.

(b) For $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to be linearly independent, the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ must have only the trivial solution. Row reducing the corresponding augmented matrix, one finds

$$\begin{bmatrix} 1 & -2 & 2 & 0 \\ -5 & 10 & -9 & 0 \\ -3 & 6 & h & 0 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, x_2 is a free variable for *any* value of h, and the homogeneous system has nontrivial solutions. Thus, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set for all h.

Note: An alternative way to approach this problem is to realize that $\mathbf{v}_1$ and $\mathbf{v}_2$ are multiples of each other: $\mathbf{v}_2 = -2\mathbf{v}_1$. Since $\mathbf{v}_3$ cannot be a multiple of either $\mathbf{v}_1$ or $\mathbf{v}_2$, regardless of h, it is not in the span of the two. For part (b), this implies that one can find the linear dependence relation $2\mathbf{v}_1 + 1\mathbf{v}_2 + 0\mathbf{v}_3 = \mathbf{0}$, i.e., a nontrivial linear combination of the zero vector, without row reduction.