Solutions to MA242 Quiz 2, 09/19/06

1. Determine if the vector $\mathbf{b}$ is a linear combination of the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} 5 \\ -6 \\ 8 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}.$$

Solution: The question is equivalent to the question if the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$ has a solution. Now, this equation has the same solution set as the linear system with augmented matrix

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{bmatrix}.$$

Row reducing this matrix, one finds

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

therefore, the linear system corresponding to the augmented matrix $[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{b}]$ is consistent, which implies that the above vector equation also has a solution. Hence, b is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.
Note: Since x_3 is a free variable, the solution $\mathbf{x}$ is *not* unique; however, uniqueness is *not* what was asked, just existence!

2. Let

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

Show that the system $A\mathbf{x} = \mathbf{b}$ is not consistent for all possible $\mathbf{b}$, and describe the set of all $\mathbf{b}$ for which the system does have a solution.

Solution: Row reducing the augmented matrix $[A\ \mathbf{b}]$, one finds

$$\begin{bmatrix} 1 & -3 & -4 & b_1 \\ -3 & 2 & 6 & b_2 \\ 5 & -1 & -8 & b_3 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & b_2 + 3b_1 \\ 0 & 0 & 0 & b_1 + 2b_2 + b_3 \end{bmatrix}.$$

Therefore, the equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $b_1 + 2b_2 + b_3 = 0$, i.e, if the rightmost column is *not* a pivot column. Hence, the system is *not* consistent for every possible choice of b; it is inconsistent if $b_1 + 2b_2 + b_3 \neq 0$.

Note: The condition that $b_1 + 2b_2 + b_3 = 0$ defines a plane through the origin in $\mathbb{R}^3$.