Solutions to MA242 Quiz 12, 12/12/06

1. Let

$$\mathbf{u}_1 = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}, \ \mathbf{u}_1 = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix}, \quad \text{and } \mathbf{y} = \begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix}.$$

(a) Verify that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set.
(b) Find the orthogonal decomposition of $\mathbf{y}$, $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, with $\hat{\mathbf{y}} \in W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ and $\mathbf{z} \in W^{\perp}$.
(c) Compute the distance from $\mathbf{y}$ to W.

Solution: (a) Since $\mathbf{u}_1 \cdot \mathbf{u}_2 = -12 + 12 = 0$, $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set.
(b) $\hat{\mathbf{y}}$ is the orthogonal projection of $\mathbf{y}$ onto W, with

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = \frac{30}{25}\mathbf{u}_1 - \frac{15}{25}\mathbf{u}_2$$
$$= \frac{6}{5}\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} - \frac{3}{5}\begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix}.$$

For $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = (0, 0, -2)$, $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, and $\hat{\mathbf{y}} \in W$, $\mathbf{z} \in W^{\perp}$ by construction.
(c) The distance from $\mathbf{y}$ to W is

$$\text{dist}(\mathbf{y}, W) = \|\mathbf{z}\| = \sqrt{0^2 + 0^2 + (-2)^2} = 2.$$

2. Given the data points $(x_1, y_1) = (1, 0)$, $(x_2, y_2) = (2, 1)$, $(x_3, y_3) = (4, 2)$, and $(x_4, y_4) = (5, 3)$, find the equation $y = \beta_0 + \beta_1 x$ of the least-squares line that best fits the data.

Solution: The design matrix X and observation vector $\mathbf{y}$ are

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}.$$

To set up the normal equations, one computes

$$X^T X = \begin{bmatrix} 4 & 12 \\ 12 & 46 \end{bmatrix} \quad \text{and} \quad X^T \mathbf{y} = \begin{pmatrix} 6 \\ 25 \end{pmatrix}.$$

Since $X^T X$ is invertible (the columns of X are clearly linearly independent), one can compute $\hat{\beta}$ via

$$\hat{\beta} = (X^T X)^{-1} X^T \mathbf{y} = \cdots = \begin{pmatrix} -0.6 \\ 0.7 \end{pmatrix},$$

and the least-squares line is given by $y = -0.6 + 0.7x$.