Solutions to MA242 Quiz 11, 12/05/06

1. Let

$$A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}.$$

(a) Solve the initial value problem $\mathbf{x}'(t) = A\mathbf{x}(t)$ for $t \geq 0$, with $\mathbf{x}(0) = (1, 1)$.
(b) Classify the nature of the origin (attractor, repellor, or saddle point).
(c) Find the directions of greatest attraction, or repulsion, and sketch typical trajectories.

Solution: (a) Since $\det(A - \lambda I_2) = \lambda^2 - 1$, the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 1$. Bases for the corresponding two eigenspaces are for instance $\mathbf{v}_1 = (-1, 1)$ and $\mathbf{v}_2 = (-3, 1)$. To find c_1 and c_2 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{x}(0)$, row reduce

$$[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{x}(0)] = \begin{bmatrix} -1 & -3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}.$$

Hence, $c_1 = 2$ and $c_2 = -1$, and the solution of the initial-value problem is given by

$$\mathbf{x}(t) = 2\begin{pmatrix} -1 \\ 1 \end{pmatrix} \mathrm{e}^{-t} + (-1)\begin{pmatrix} -3 \\ 1 \end{pmatrix} \mathrm{e}^{t}.$$

(b) Since one of the eigenvalues is negative and the other one is positive, the origin is a saddle point of $\mathbf{x}' = A\mathbf{x}$.
(c) The direction of greatest attraction is the line through $\mathbf{v}_1$ and the origin; the direction of greatest repulsion is the line through $\mathbf{v}_2$ and the origin.

2. Let the vectors $\mathbf{u}$ and $\mathbf{v}$ be defined by

$$\mathbf{u} = \begin{pmatrix} 1 \\ -2 \\ 3 \\ -7 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} -3 \\ 5 \\ 4 \\ 0 \end{pmatrix}.$$

(a) Determine if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
(b) Compute the distance $\mathrm{dist}(\mathbf{u}, \mathbf{v})$ between $\mathbf{u}$ and $\mathbf{v}$.
(c) Find a unit vector in the direction of $\mathbf{u}$.

Solution: (a) Since $\mathbf{u} \cdot \mathbf{v} = 1(-3) + (-2)5 + 3(4) + (-7)0 = -1 \neq 0$, $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal.
(b) Since $\mathbf{u} - \mathbf{v} = (4, -7, -1, -7)$,

$$\mathrm{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{4^2 + (-7)^2 + (-1)^2 + (-7)^2} = \sqrt{115} \approx 10.72.$$

(c) Since $\|\mathbf{u}\| = \sqrt{1^2 + (-2)^2 + 3^2 + (-7)^2} = \sqrt{63} = 3\sqrt{7}$, a unit vector in the direction of $\mathbf{u}$ is given by

$$\frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{3\sqrt{7}}\begin{pmatrix} 1 \\ -2 \\ 3 \\ -7 \end{pmatrix}.$$