Solutions to MA242 Quiz 10, 11/28/06

1. Define the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix}.$$

Find a basis $\mathcal{B}$ for $\mathbb{R}^2$ with the property that $[T]_{\mathcal{B}}$ is diagonal.

Solution: Diagonalize A by finding the eigenvalues and eigenvectors of A: The characteristic polynomial is $\lambda^2 - 5\lambda = \lambda(\lambda - 5)$, so the eigenvalues of A are 5 and 0. For $\lambda = 5$,

$$A - 5I_2 = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix},$$

and a basis for the eigenspace is thus $\mathbf{v}_1 = (-2, 1)$. Similarly, for $\lambda = 0$, a basis is given by a solution to $A\mathbf{x} = \mathbf{0}$; one can take, e.g., $\mathbf{v}_2 = (3, 1)$. From $\mathbf{v}_1$ and $\mathbf{v}_2$, we construct

$$P = [\mathbf{v}_1 \ \mathbf{v}_2] = \begin{bmatrix} -2 & 3 \\ 1 & 1 \end{bmatrix}.$$

By Theorem 8, the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ has the property that the $\mathcal{B}$-matrix of $\mathbf{x} \mapsto A\mathbf{x}$ is diagonal.

2. Define the matrix C by

$$C = \begin{bmatrix} \sqrt{3} & 3 \\ -3 & \sqrt{3} \end{bmatrix}.$$

(a) Show that the eigenvalues of C are $\sqrt{3} \pm 3i$, with corresponding eigenvectors $(1, \pm i)$.

(b) The transformation $\mathbf{x} \mapsto C\mathbf{x}$ is the composition of a rotation and a scaling. Give the angle φ of the rotation $(-\pi \le \varphi \le \pi)$, and give the scaling factor r.

Solution: (a) Computing

$$C\mathbf{x} = \begin{bmatrix} \sqrt{3} & 3 \\ -3 & \sqrt{3} \end{bmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \sqrt{3} + 3i \\ -3 + \sqrt{3}i \end{pmatrix} = (\sqrt{3} + 3i) \begin{pmatrix} 1 \\ i \end{pmatrix},$$

one sees that $(1, i)$ is an eigenvector corresponding to the eigenvalue $\sqrt{3} + 3i$. Therefore, the complex conjugate vector $(1, -i)$ must also be an eigenvector corresponding to $\sqrt{3} - 3i$.

(b) The angle of rotation of the transformation is

$$\varphi = \arctan\left(\frac{-3}{\sqrt{3}}\right) = -\frac{\pi}{3};$$

the scaling factor is

$$r = |\lambda| = \sqrt{(\sqrt{3})^2 + (-3)^2} = 2\sqrt{3}.$$