MA 242 LINEAR ALGEBRA C1, Solutions to Second Midterm Exam

Prof. Nikola Popovic, November 9, 2006, 09:30am - 10:50am

Problem 1 (15 points).
Let the matrix A be given by

$$\begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}.$$

(a) Find the inverse A^{-1} of A, if it exists.

(b) Based on your answer in (a), determine whether the columns of A span $\mathbb{R}^3$. (*Justify* your answer!)

Solution.

(a) To check whether A is invertible, we row reduce the augmented matrix $[A\ I_3]$:

$$\begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & 0 & -7 & -2 & 1 \end{bmatrix}.$$

Since the last row in the echelon form of A contains only zeros, A is not row equivalent to I_3. Hence, A is not invertible, and A^{-1} does not exist.

(b) Since A is not invertible by (a), the Invertible Matrix Theorem says that the columns of A cannot span $\mathbb{R}^3$.

Problem 2 (15 points).
Let the vectors $\mathbf{b}_1, \ldots, \mathbf{b}_4$ be defined by

$$\mathbf{b}_1 = \begin{pmatrix} 3 \\ 5 \\ -2 \\ 4 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 2 \\ -1 \\ -5 \\ 7 \end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix} -1 \\ 1 \\ 3 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{b}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -3 \end{pmatrix}.$$

(a) Determine if the set $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ is linearly independent by computing the determinant of the matrix $B = [\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3\ \mathbf{b}_4]$.

(b) Using your answer in (a), determine if $\mathcal{B}$ is a basis for $\mathbb{R}^4$. (*Justify* your answer!)

Solution.

(a) The determinant of B is most easily computed by first going down the fourth column,

$$\det B = \det \begin{bmatrix} 3 & 2 & -1 & 0 \\ 5 & -1 & 1 & 0 \\ -2 & -5 & 3 & 0 \\ 4 & 7 & 0 & -3 \end{bmatrix} = (-1)^{4+4}(-3) \det \underbrace{\begin{bmatrix} 3 & 2 & -1 \\ 5 & -1 & 1 \\ -2 & -5 & 3 \end{bmatrix}}_{B_{44}}.$$

Now, one possibility to compute the determinant of the 3×3-submatrix B_{44} is

$$\begin{aligned} \det B_{44} &= (3)(-1)(3) + (2)(1)(-2) + (-1)(5)(-5) \\ &\quad - [(-2)(-1)(-1) + (-5)(1)(3) + (3)(5)(2)] = -1. \end{aligned}$$

Hence, $\det B = (-3)(-1) = 3$.

(b) Since $\det B \neq 0$, it follows that the matrix B is invertible. Hence, by the Invertible Matrix Theorem, the columns of B are linearly independent, and the columns of B span $\mathbb{R}^4$. Therefore, the set $\mathcal{B}$ is a basis for $\mathbb{R}^4$.

Problem 3 (15 points).
Let the matrix A be given by

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

(a) Find a basis for the column space Col$A$ of A.

(b) Find a basis for the null space Nul$A$ of A.

(c) What are the dimensions of Col$A$ and Nul$A$? (*Justify* your answers!)

Solution.

(a) To find a basis for Col$A$, we have to reduce A to echelon form:

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The pivot columns in the echelon form are the first and third columns; therefore, a basis for Col$A$ is given by the first and third columns of A, $(-3, 1, 2)$ and $(-1, 2, 5)$.

(b) To obtain a basis for Nul$A$, we have to find the solution set of $A\mathbf{x} = \mathbf{0}$. Hence, we continue row reducing until the reduced echelon form of A is found:

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The basic variables x_1 and x_3 can be expressed in terms of the free variables x_2, x_4, and x_5, with $x_1 = 2x_2 + x_4 - 3x_5$ and $x_3 = -2x_4 + 2x_5$. The general solution in parametric vector form is given by

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

Hence, a basis for Nul$A$ is given by the three vectors $(2, 1, 0, 0, 0)$, $(1, 0, -2, 1, 0)$, and $(-3, 0, 2, 0, 1)$.

(c) Since the basis for Col$A$ found in (a) consists of two vectors, the dimension of Col$A$ is 2. Since the basis for Nul$A$ found in (b) consists of three vectors, the dimension of Nul$A$ is 3.

(*Note:* This agrees with the Rank Theorem, since dim(Col$A$)+dim(Nul$A$)= 5 equals the number of columns of A.)

Problem 4 (15 points).
Let V be a vector space, and let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for V. Show that the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is an isomorphism from V onto $\mathbb{R}^n$.

Solution.

To show that the coordinate mapping is an isomorphism, we have to show that it is linear, one-to-one, and onto. For vectors $\mathbf{x}$ and $\mathbf{y}$ in V, let $\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$ and $\mathbf{y} = d_1\mathbf{b}_1 + \ldots + d_n\mathbf{b}_n$. Then, $[\mathbf{x}]_{\mathcal{B}} = (c_1, \ldots, c_n)$ and $[\mathbf{y}]_{\mathcal{B}} = (d_1, \ldots, d_n)$. Moreover, $\mathbf{x} + \mathbf{y} = (c_1 + d_1)\mathbf{b}_1 + \ldots + (c_n + b_n)\mathbf{b}_n$, and

$$\begin{aligned} [\mathbf{x} + \mathbf{y}]_{\mathcal{B}} &= (c_1 + d_1, \ldots, c_n + d_n) = (c_1, \ldots, c_n) + (d_1, \ldots, d_n) \\ &= [\mathbf{x}]_{\mathcal{B}} + [\mathbf{y}]_{\mathcal{B}}. \end{aligned}$$

Also, $c\mathbf{x} = (cc_1)\mathbf{b}_1 + \ldots + (cc_n)\mathbf{b}_n$, and

$$\begin{aligned} [c\mathbf{x}]_{\mathcal{B}} &= (cc_1, \ldots, cc_n) = c(c_1, \ldots, c_n) \\ &= c[\mathbf{x}]_{\mathcal{B}}, \end{aligned}$$

and the coordinate mapping is therefore linear. To show that it is one-to-one, assume that $[\mathbf{x}]_{\mathcal{B}} = (c_1, \ldots, c_n) = [\mathbf{y}]_{\mathcal{B}}$ for two vectors $\mathbf{x}$ and $\mathbf{y}$ in V. Then,

$$\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n \text{ and } \mathbf{y} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n,$$

so $\mathbf{x} = \mathbf{y}$, which proves one-to-one-ness. To show that the coordinate mapping is onto, let $(c_1, \ldots, c_n)$ be a vector in $\mathbb{R}^n$. Then, $(c_1, \ldots, c_n)$ is the image of the vector $\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$ in V, that is, $[\mathbf{x}]_{\mathcal{B}} = (c_1, \ldots, c_n)$, which proves onto-ness.

(*Note:* To prove that V and $\mathbb{R}^n$ are isomorphic for *general* vector spaces V, you *cannot* use the change-of-coordinates matrix $P_{\mathcal{B}} = [\mathbf{b}_1 \ \ldots \ \mathbf{b}_n]$. This matrix is only defined if $V = \mathbb{R}^n$, that is, if the vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are vectors in $\mathbb{R}^n$ that can be written into a matrix!)

Problem 5 (20 points).
Let $\mathbb{P}$ denote the vector space of all polynomials, and let $\mathbb{P}_2$ be the set of all polynomials of degree at most 2; that is, $\mathbb{P}_2 = \{\mathbf{p}(t) : \ \mathbf{p}(t) = a_0 + a_1 t + a_2 t^2, \ a_0, a_1, a_2 \text{ real}\}$.

(a) Show that $\mathbb{P}_2$ is a subspace of $\mathbb{P}$.

(b) Using coordinate vectors, show that the set $\mathcal{B}$ given by

$$\mathcal{B} = \{1 + t^2, 2 - t + 3t^2, 1 + 2t - 4t^2\}$$

is a basis for $\mathbb{P}_2$.

(c) Find the coordinate vector $[\mathbf{p}]_{\mathcal{B}}$ of the polynomial $\mathbf{p}(t) = -4 - t^2$ relative to $\mathcal{B}$.

(d) Find the polynomial $\mathbf{q}(t)$ whose coordinate vector relative to $\mathcal{B}$ is $[\mathbf{q}]_{\mathcal{B}} = (-3, 1, 2)$.

Solution.

(a) Since the zero polynomial $\mathbf{p} = \mathbf{0}$ is obtained for $a_0 = a_1 = a_2 = 0$, $\mathbb{P}_2$ contains the zero vector of $\mathbb{P}$. Given two polynomials $\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2$ and $\mathbf{q}(t) = b_0 + b_1 t + b_2 t^2$ in $\mathbb{P}_2$, the sum

$$(\mathbf{p} + \mathbf{q})(t) = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2$$

is in $\mathbb{P}_2$. Hence, $\mathbb{P}_2$ is closed under vector addition. Also, for any scalar c,

$$(c\mathbf{p})(t) = (ca_0) + (ca_1)t + (ca_2)t^2$$

is in $\mathbb{P}_2$, and $\mathbb{P}_2$ is closed under scalar multiplication. So, in sum, $\mathbb{P}_2$ is a subspace of $\mathbb{P}$.

(b) The coordinate vectors of the polynomials $1 + t^2$, $2 - t + 3t^2$, and $1 + 2t - 4t^2$ are $(1, 0, 1)$, $(2, -1, 3)$, and $(1, 2, -4)$, respectively. (The entries in the coordinate vectors contain the coefficients of 1, t, and t^2, respectively.) Since the matrix formed from these vectors is row equivalent to the identity matrix I_3,

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 3 & -4 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -3 \end{bmatrix} \sim \ldots \sim I_3,$$

the coordinate vectors are linearly independent and span $\mathbb{R}^3$. By the isomorphism between $\mathbb{P}_2$ and $\mathbb{R}^3$, the corresponding polynomials $1 + t^2$, $2 - t + 3t^2$, and $1 + 2t - 4t^2$ are linearly independent and span $\mathbb{P}_2$. Therefore, they form a basis for $\mathbb{P}_2$.

(c) To find $[\mathbf{p}]_{\mathcal{B}}$, we have to determine how $\mathbf{p}(t) = -4 - t^2 = (-4)1 + (0)t + (-1)t^2$ can be combined from the polynomials in $\mathcal{B}$. This can be done by solving the linear system obtained from the corresponding coordinate vectors:

$$\begin{bmatrix} 1 & 2 & 1 & -4 \\ 0 & -1 & 2 & 0 \\ 1 & 3 & -4 & -1 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Hence, $[\mathbf{p}]_{\mathcal{B}} = (1, -2, 1)$.

(d) To find the polynomial $\mathbf{q}$ corresponding to $[\mathbf{q}]_{\mathcal{B}} = (-3, 1, 2)$, we just compute the matrix-vector product

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 3 & -4 \end{bmatrix} \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -8 \end{pmatrix}.$$

Therefore, $\mathbf{q}(t) = 1 + 3t - 8t^2$.

Problem 6 (20 points).
Determine whether the statements below are true or false. (*Justify* your answers: If a statement is true, explain why it is true; if it is false, explain why, or give a counter-example for which it is false.)

(a) If A and B are $m \times n$-matrices, then both AB^T and $A^T B$ are defined.

(b) The determinant of an $n \times n$-matrix A is the product of the diagonal entries in A.

(c) For A an $m \times n$-matrix, Col$A$ is the set of all solutions of the linear system $A\mathbf{x} = \mathbf{b}$.

(d) For $\mathbf{x}$ in $\mathbb{R}^n$, the coordinate vector $[\mathbf{x}]_{\mathcal{E}}$ of $\mathbf{x}$ relative to the standard basis $\mathcal{E}$ is $\mathbf{x}$ itself.

Solution.

(a) True. For A and B $m \times n$, A^T and B^T are $n \times m$, so both matrix products AB^T and $A^T B$ are defined: The number of columns of A, n, equals the number of rows of B^T; the number of columns of A^T, m, equals the number of rows of B.

(b) False; this statement is only true if A is a triangular matrix. Take e.g.

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix};$$

then, det$A$ = (1)(0) – (1)(2) $= -2$, whereas the product of the diagonal elements is 0.

(c) False. The column space Col$A$ is the set of all linear combinations of the columns of A, that is,

$$\text{Col}A = \{\mathbf{b} \text{ in } \mathbb{R}^m : \ A\mathbf{x} = \mathbf{b} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}.$$

The solution set of $A\mathbf{x} = \mathbf{b}$ would be the set of all $\mathbf{x}$ in $\mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{b}$!

(d) True. In general, the coordinate vector of $\mathbf{x}$ in $\mathbb{R}^n$ relative to a basis $\mathcal{B}$ is related to $\mathbf{x}$ by $P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}$. For $\mathcal{B} = \mathcal{E}$, the change-of-coordinates matrix $P_{\mathcal{E}}$ is the identity matrix I_n. Hence, $\mathbf{x} = I_n[\mathbf{x}]_{\mathcal{E}} = [\mathbf{x}]_{\mathcal{E}}$.