MA 242 LINEAR ALGEBRA C1, Solutions to First Midterm Exam

Prof. Nikola Popovic, October 5, 2006, 09:30am - 10:50am

Problem 1 (15 points).
Determine h and k such that the solution set of

$$\begin{aligned} x_1 + 3x_2 &= k \\ 4x_1 + hx_2 &= 8 \end{aligned}$$

(a) is empty, (b) contains a unique solution, and (c) contains infinitely many solutions. (Give separate answers for each part, and *justify* them.)

Solution.
Row reducing the augmented matrix of the linear system, we find

$$\begin{bmatrix} 1 & 3 & k \\ 4 & h & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & k \\ 0 & h-12 & 8-4k \end{bmatrix}.$$

Hence, we have the following possibilities:

(a) The system is inconsistent (i.e., the solution set is empty) if the third column of the augmented matrix contains a pivot. This is the case if $h - 12 = 0$ and $8 - 4k \neq 0$, that is, if $h = 12$ and $k \neq 2$.

(b) The system has a unique solution if the first two columns contain pivots, that is, if both x_1 and x_2 are basic variables. Hence, $h \neq 12$ must hold for the second column to contain a pivot. Note that k is arbitrary.

(c) The system has infinitely many solutions if x_2 is a free variable and if the third column does not contain a pivot, which is the case for $h = 12$ and $k = 2$.

Problem 2 (15 points).
Determine if the following vectors are linearly dependent or linearly independent. (*Justify* your answers.)

(a) $\left\{ \begin{pmatrix} -4 \\ 0 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 3 \\ 6 \end{pmatrix} \right\}$, (b) $\left\{ \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 8 \\ 1 \end{pmatrix} \right\}$,

(c) $\left\{ \begin{pmatrix} -8 \\ 12 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \right\}$.

Solution.

(a) The set is linearly dependent, since it contains the zero vector $\mathbf{0}$. (Any set containing the zero vector is linearly dependent.)

(b) The set is linearly dependent, since it consists of four vectors in $\mathbb{R}^2$. (Any set containing more vectors than each vector has entries is linearly dependent.)

(c) The set is linearly independent, since it contains two vectors neither of which is the zero vector, and since these vectors are not multiples of each other.

Problem 3 (15 points).
Let the matrix A be given by

$$\begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix},$$

(a) Determine whether the columns of A span $\mathbb{R}^4$.

(b) Based on your answer in (a), decide whether the matrix equation $A\mathbf{x} = \mathbf{b}$ has a solution for *every* $\mathbf{b}$ in $\mathbb{R}^4$. (*Justify* your answer.)

Solution.

(a) To determine whether the columns of A span $\mathbb{R}^4$, we row reduce A,

$$\begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & -1 & -1 & 5 \\ 0 & -2 & -2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 \end{bmatrix}.$$

The row reduced matrix contains three pivot positions and therefore does not have a pivot in every row. Hence, the columns of A do not span $\mathbb{R}^4$, see Theorem 4.

(b) No, $A\mathbf{x} = \mathbf{b}$ does not have a solution for every $\mathbf{b}$ in $\mathbb{R}^4$, since that would be equivalent to the columns of A spanning $\mathbb{R}^4$, again by Theorem 4.

Problem 4 (20 points).
Consider the following system of linear equations,

$$\begin{aligned} x_1 - 3x_2 - 4x_3 &= b_1 \\ -3x_1 + 3x_2 + 6x_3 &= b_2 \\ 5x_1 - 3x_2 - 8x_3 &= b_3, \end{aligned}$$

where b_1, b_2, and b_3 are real numbers.

(a) Write the system first as a vector equation and then as a matrix equation.

(b) Show that the system is *not* consistent for all possible $\mathbf{b} = (b_1, b_2, b_3)$, and describe the set of all $\mathbf{b}$ for which it *does* have a solution.

(c) Find the solution sets of the system for $\mathbf{b} = (0, 0, 0)$ and for $\mathbf{b} = (2, 0, -2)$. Write them in parametric vector form; then, illustrate and compare them geometrically.

Solution.

(a) The equivalent vector equation is

$$x_1 \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 3 \\ -3 \end{pmatrix} + x_3 \begin{pmatrix} -4 \\ 6 \\ -8 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

while the corresponding matrix equation is

$$\begin{bmatrix} 1 & -3 & -4 \\ -3 & 3 & 6 \\ 5 & -3 & -8 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

(b) To determine for which choices of $\mathbf{b} = (b_1, b_2, b_3)$ the system is consistent, we row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & -4 & b_1 \\ -3 & 3 & 6 & b_2 \\ 5 & -3 & -8 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -6 & -6 & b_2 + 3b_1 \\ 0 & 12 & 12 & b_3 - 5b_1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -6 & -6 & b_2 + 3b_1 \\ 0 & 0 & 0 & b_3 - 5b_1 + 2b_2 + 6b_1 \end{bmatrix}.$$

The system is inconsistent if the last column contains a pivot, that is, if $b_3 - 5b_1 + 2b_2 + 6b_1 = b_1 + 2b_2 + b_3 \neq 0$. The set of all $\mathbf{b}$ for which the system *is* consistent is described by $b_1 + 2b_2 + b_3 = 0$, which is a plane in $\mathbb{R}^3$.

(c) To explicitly find solutions of the system for specific choices of $\mathbf{b}$, we reduce the augmented matrix to reduced echelon form:

$$\begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -6 & -6 & b_2 + 3b_1 \\ 0 & 0 & 0 & b_1 + 2b_2 + b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & 1 & 1 & -\frac{3b_1+b_2}{6} \\ 0 & 0 & 0 & b_1 + 2b_2 + b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & b_1 - \frac{3b_1+b_2}{2} \\ 0 & 1 & 1 & -\frac{b_1+3b_2}{6} \\ 0 & 0 & 0 & b_1 + 2b_2 + b_3 \end{bmatrix}.$$

For $\mathbf{b} = (0, 0, 0)$, this matrix becomes

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

the general solution is given by $x_1 = x_3$, $x_2 = -x_3$, and x_3 free. In parametric vector form, this corresponds to

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix},$$

which defines a line in $\mathbb{R}^3$ through the origin. For $\mathbf{b} = (2, 0, -2)$, the augmented row reduced matrix becomes

$$\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

the general solution is $x_1 = -1 + x_3$, $x_2 = -1 - x_3$, and x_3 free. In parametric vector form, this can be written as

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 + x_3 \\ -1 - x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix},$$

which is a line through the point $(-1, -1, 0)$. Hence, the solution sets of the homogeneous system and the nonhomogeneous system are parallel lines in $\mathbb{R}^3$ offset by the vector $(-1, -1, 0)$.

Problem 5 (15 points).
Let the transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$, $\mathbf{x} \mapsto T(\mathbf{x})$ be defined by

$$T(\mathbf{x}) = T(x_1, x_2) = (-3x_1 + 2x_2, x_1 - 4x_2, x_2).$$

(a) Determine whether T is linear, and find the standard matrix of T.

(b) Determine whether T is one-to-one and onto. (*Justify* your answers.)

Solution.

(a) Since

$$T(x_1, x_2) = \begin{pmatrix} -3x_1 + 2x_2 \\ x_1 - 4x_2 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = \begin{bmatrix} -3 & 2 \\ 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

the standard matrix of T is precisely

$$A = \begin{bmatrix} -3 & 2 \\ 1 & -4 \\ 0 & 1 \end{bmatrix}.$$

Therefore, also, T must be linear, since every transformation that can be described by a matrix is linear.

(b) T is one-to-one, since the columns of A are linearly independent (they are non-zero, and not multiples of each other). T is not onto, since the columns of A do not span $\mathbb{R}^3$. (Alternatively, T is one-to-one since A contains a pivot in every column; however, it is not onto, since A does not contain a pivot in every row.)

Problem 6 (20 points).
Determine whether the statements below are true or false. (*Justify* your answers: If a statement is true, explain why it is true; if it is false, explain why, or give a counter-example for which it is false.)

(a) Two linear systems are equivalent if they have the same solution set.

(b) Not every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation.

(c) A linear combination of vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ can always be written in the form $A\mathbf{x}$ for a suitable matrix A and vector $\mathbf{x}$.

(d) The echelon form of a matrix is unique.

Solution.

(a) True, by the definition of equivalent systems.

(b) False. Every linear transformation *from* $\mathbb{R}^n$ *to* $\mathbb{R}^m$ is a matrix transformation. (The corresponding matrix is the standard matrix of the transformation.)

(c) True, by the definition of the matrix-vector product:

$$x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n = [\mathbf{a}_1 \ldots \mathbf{a}_n] \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

with $A = [\mathbf{a}_1 \ldots \mathbf{a}_n]$.

(d) False, the *reduced* echelon form of a matrix is unique. For example, the matrices

$$\begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

are both echelon forms of

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$