MA 124 CALCULUS II C1, Solutions to Second Midterm Exam

Prof. Nikola Popovic, March 30, 2006, 08:00am - 09:20am

Problem 1 (15 points).
Let $C(t)$ be the concentration of a drug in the bloodstream. As the body eliminates the drug, $C(t)$ decreases at a rate that is proportional to the amount of drug that is present at the time, i.e.,

$$\frac{dC}{dt} = -kC \qquad \text{with } k > 0 \text{ constant.}$$

(a) If C_0 is the concentration given at time $t = 0$, find the concentration at time t.

(b) Find the value of k if the initial concentration is $C_0 = 50$ and if $C = 25$ at time $t = 30$.

Solution.

(a) We know that the solution to the initial value problem $\frac{dC}{dt} = -kC$, $C(0) = C_0$ is given by $C(t) = C_0 \cdot \mathrm{e}^{-kt}$. (This is the "law of natural decay", and follows from the fact that the differential equation is separable:

$$\begin{aligned} \frac{dC}{C} &= -k\,dt \\ \int \frac{dC}{C} &= -\int k\,dt \\ \ln|C| &= -kt + A \\ |C| &= \mathrm{e}^{-kt+A} = \mathrm{e}^{A} \cdot \mathrm{e}^{-kt} \\ C(t) &= B \cdot \mathrm{e}^{-kt} \qquad (\text{with } B = \pm \mathrm{e}^{A}). \end{aligned}$$

To fix B, note that $C(0) = C_0$ by assumption; therefore, $C_0 = B \cdot \mathrm{e}^{-k0} = B$.)

(b) For $C_0 = 50$, we find with (a) that the solution is given by $C(t) = 50 \cdot \mathrm{e}^{-kt}$. To fix k, we have to use the fact that $C(30) = 25$, i.e., that the solution passes through the point $(t, C) = (30, 25)$. Hence,

$$\begin{aligned} 25 &= 50 \cdot \mathrm{e}^{-k30} \\ \frac{1}{2} &= \mathrm{e}^{-k30} \\ \ln\frac{1}{2} &= -k30 \\ k &= -\frac{1}{30}\ln\frac{1}{2} = \frac{1}{30}\ln 2 \approx 0.0231. \end{aligned}$$

Problem 2 (15 points).
Find the solution of the differential equation

$$\frac{dx}{dt} = (1-t)\cdot(1+x)$$

that passes through the point $(t, x) = (0, 0)$.

Solution. The differential equation is separable, since the right-hand side is the product of a function of t (the independent variable) times a function of x (the dependent variable). Hence,

$$\begin{aligned} \frac{dx}{1+x} &= (1-t)\,dt \\ \int \frac{dx}{1+x} &= \int (1-t)\,dt \\ \ln|1+x| &= t - \frac{t^2}{2} + C \\ |1+x| &= \mathrm{e}^{t-\frac{t^2}{2}+C} = \mathrm{e}^C \cdot \mathrm{e}^{t-\frac{t^2}{2}} \\ 1+x &= A \cdot \mathrm{e}^{t-\frac{t^2}{2}} \qquad \text{(with } A = \pm \mathrm{e}^C\text{)} \\ x(t) &= A \cdot \mathrm{e}^{t-\frac{t^2}{2}} - 1. \end{aligned}$$

This is the general solution of the equation. To find the solution which passes through the point $(0,0)$, we have to fix the constant A:

$$x(0) = 0 = A \cdot \mathrm{e}^{0-\frac{0^2}{2}} - 1 = A - 1.$$

Therefore, $A = 1$, and the unique solution for which $x(0) = 0$ is given by $x(t) = \mathrm{e}^{t-\frac{t^2}{2}} - 1$.

Problem 3 (20 points).
Let

$$f(x) = \begin{cases} \dfrac{\pi}{20} \sin\left(\dfrac{\pi x}{10}\right) & \text{if } 0 \le x \le 10, \\ 0 & \text{if } x < 0 \text{ or } x > 10. \end{cases}$$

(a) Show that $f(x)$ is a probability density function.

(b) Find $P(X < 4)$.

Solution.

(a) To show that $f(x)$ is a probability density function, we have to show

$$f(x) \ge 0 \quad \text{for all } x \qquad \text{and} \qquad \int_{-\infty}^{\infty} f(x)\,dx = 1.$$

Since $f(x) = 0$ for $x < 0$ or $x > 10$, we only have to consider $x \in [0, 10]$. To show that $f(x) \ge 0$ there, note that for $0 \le x \le 10$, there holds $0 \le \frac{\pi x}{10} \le \pi$. Since the sine function is positive on the interval $(0, \pi)$ and zero at 0 and at π and since $\frac{\pi}{20} > 0$, it follows that $f(x) \ge 0$ for $0 \le x \le 10$. To show that the second requirement holds, we compute

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{10} \frac{\pi}{20} \sin\left(\frac{\pi x}{10}\right) dx = \frac{\pi}{20} \int_0^{\pi} \frac{10}{\pi} \sin(u)\,du = \frac{1}{2}\big(-\cos(u)\big)\Big|_0^{\pi} = 1.$$

(Here, we have made the substitution $u = \frac{\pi}{10}x$, $du = \frac{\pi}{10}dx$ to evaluate the integral.) Therefore, $f(x)$ is a probability density function.

(b) Since $f(x) = 0$ for $x < 0$, we have

$$P(X < 4) = \int_0^4 \frac{\pi}{20} \sin\left(\frac{\pi x}{10}\right) dx = \frac{1}{2}\big(-\cos(u)\big)\Big|_0^{\frac{2\pi}{5}} \approx 0.3455.$$

Hence, the probability that X is less than 4 is about 34.55%.

Problem 4 (15 points).
Let the curve C be defined by

$$y(x) = \int_1^x \sqrt{\sqrt{t} - 1}\,dt \qquad \text{for } 1 \le x \le 16.$$

Find the length L of C. $\Big($*Hint:* Apply the Fundamental Theorem of Calculus to find $\dfrac{dy}{dx}$.$\Big)$

Solution. The curve C is parametrized by x, with $x \in [1, 16]$. Hence, to find the length of C, we make use of the formula

$$L = \int_1^{16} \sqrt{\Big(\frac{dy}{dx}\Big)^2 + 1}\,dx.$$

To compute $\frac{dy}{dx}$, we apply the Fundamental Theorem of Calculus:

$$\frac{dy}{dx} = \frac{d}{dx}\Big(\int_1^x \sqrt{\sqrt{t} - 1}\,dt\Big) = \sqrt{\sqrt{x} - 1}.$$

Therefore, $(\frac{dy}{dx})^2 = \sqrt{x} - 1$, and

$$L = \int_1^{16} \sqrt{\sqrt{x} - 1 + 1}\,dx = \int_1^{16} x^{\frac{1}{4}}\,dx = \frac{4}{5} \cdot x^{\frac{5}{4}}\Big|_1^{16} = \frac{124}{15} = 24.8.$$

Problem 5 (20 points).
Let the sequence $\{a_n\}$ be defined by $a_n = \dfrac{2n - 3}{3n + 4}$.

(a) Determine whether the sequence is increasing, decreasing, or not monotonic.

(b) Determine whether the sequence is bounded.

Solution.

(a) Computing the first four terms of the sequence, we find $a_1 = -\frac{1}{7} \approx -0.1429$, $a_2 = \frac{1}{10} = 0.1$, $a_3 = \frac{3}{13} \approx 0.2308$, and $a_4 = \frac{5}{16} = 0.3125$. Hence, we guess that $\{a_n\}$ is increasing. To prove our guess, we write

$$\begin{aligned} a_n &< a_{n+1} \\ \Leftrightarrow \frac{2n-3}{3n+4} &< \frac{2(n+1)-3}{3(n+1)+4} \\ \Leftrightarrow (2n-3)\cdot(3n+7) &< (2n-1)\cdot(3n+4) \qquad \text{(by cross-multiplication)} \\ \Leftrightarrow 6n^2 - 9n + 14n - 21 &< 6n^2 - 3n + 8n - 4 \\ \Leftrightarrow -21 &< -4. \end{aligned}$$

Since the last statement is true, all the preceding statements are also true (as they are all equivalent). Hence, $a_n < a_{n+1}$, and we have proved that $\{a_n\}$ is increasing. (An alternative way to show that $\{a_n\}$ is increasing is to consider the corresponding function $f(x) = \frac{2x-3}{3x+4}$, and to prove that $f(x)$ is increasing for $x > 0$, i.e., that $f'(x) > 0$ holds:

$$f'(x) = \frac{2\cdot(3x+4) - (2x-3)\cdot 3}{(3x+4)^2} = \frac{17}{(3x+4)^2} > 0,$$

and since $a_n = f(n)$, the sequence $\{a_n\}$ is also increasing.)

(b) For $\{a_n\}$ to be bounded, it has to be bounded both *above* and *below*. Since $\{a_n\}$ is increasing, the smallest term in the sequence is the first term a_1. Hence, the sequence is bounded below by $a_1 = -\frac{1}{7}$, i.e., $a_n \geq -\frac{1}{7}$ for every $n \geq 1$. To show that the sequence is bounded above, we estimate

$$a_n = \frac{2n-3}{3n+4} < \frac{2n-3}{3n} < \frac{2n}{3n} = \frac{2}{3} \qquad \text{for every } n \geq 1.$$

Hence, the sequence is bounded above by $\frac{2}{3}$. (Alternatively, one can argue that $\lim\limits_{n\to\infty} a_n = \frac{2}{3}$ and, hence, that the sequence is bounded above by its limit, since it is increasing and approaching that limit from below.)

Problem 6 (15 points).

Determine whether the statements below are true or false. If a statement is true, explain why; if it is false, explain why or give an example that disproves the statement.

(a) The function $f(x) = \dfrac{\ln x}{x}$ is a solution of the differential equation $x^2y' + xy = 1$.

(b) The differential equation $\dfrac{dy}{dx} = x - 2y$ is separable.

(c) Every monotonic sequence is convergent.

Solution.

(a) TRUE. To verify that $y = f(x) = \frac{\ln x}{x}$ is a solution of the given equation, we have to plug y as well as y' into the equation and see whether it is satisfied. Compute

$$y' = f'(x) = \frac{\frac{1}{x}\cdot x - \ln x \cdot 1}{x^2} = \frac{1-\ln x}{x^2};$$

then,

$$x^2y' + xy = x^2 \cdot \frac{1-\ln x}{x^2} + x \cdot \frac{\ln x}{x} = 1 - \ln x + \ln x = 1,$$

as required.

(b) FALSE. For a differential equation of the form $\frac{dy}{dx} = F(x,y)$ to be separable, we have to be able to write its right-hand side as a *product* of two functions $f(x)$ and $g(y)$ which only depend on x and y, respectively: $\frac{dy}{dx} = f(x)\cdot g(y)$. In our case, however, the right-hand side can only be written as the *difference* of two such functions.

(c) FALSE. A sequence $\{a_n\}$ can be monotonic (i.e., increasing or decreasing) but not converge to any limit if it increases or decreases without bound, i.e., if $\lim\limits_{n\to\infty} a_n = \pm\infty$. Counter-examples are e.g. $a_n = n$ (increasing, but $\lim\limits_{n\to\infty} n = \infty$) or $a_n = -\mathrm{e}^n$ (decreasing, but $\lim\limits_{n\to\infty}(-\mathrm{e}^n) = -\infty$).