MA 124 CALCULUS II C1, Solutions to First Midterm Exam

Prof. Nikola Popovic, February 16, 2006, 08:00am - 09:20am

Problem 1 (15 points).
Determine whether the statements below are true or false. If a statement is true, explain why; if it is false, give a counter-example.

(a) If f and g are continuous on $[a, b]$, then

$$\int_a^b [f(x) \cdot g(x)]\,dx = \left(\int_a^b f(x)\,dx\right) \cdot \left(\int_a^b g(x)\,dx\right).$$

(b) If f is a continuous, decreasing function on $[a, \infty)$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_a^\infty f(x)\,dx$ is convergent.

(c) All continuous functions have antiderivatives.

Solution.

(a) FALSE. Take e.g. $f(x) = x$, $g(x) = x$, and $[a, b] = [0, 1]$. Then,

$$\int_a^b [f(x) \cdot g(x)]\,dx = \int_0^1 x \cdot x\,dx = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3},$$

but

$$\left(\int_a^b f(x)\,dx\right) \cdot \left(\int_a^b g(x)\,dx\right) = \left(\int_0^1 x\,dx\right) \cdot \left(\int_0^1 x\,dx\right) = \frac{x^2}{2}\Big|_0^1 \cdot \frac{x^2}{2}\Big|_0^1 = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

(b) FALSE. Take e.g. $f(x) = \frac{1}{x}$ and $a = 1$; then, $f(x)$ is continuous on $[1, \infty)$, $\lim_{x\to\infty} f(x) = 0$, and $f(x)$ is decreasing on $[1, \infty)$ (since $f'(x) = -\frac{1}{x^2} < 0$), but

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln x\Big|_1^t = \lim_{t\to\infty} \ln t = \infty.$$

Hence, the improper integral diverges.

(c) TRUE. By the Fundamental Theorem of Calculus, Part I, we know that if $f(x)$ is continuous on $[a, b]$,

$$g(x) = \int_a^x f(t)\,dt$$

is an antiderivative of f, i.e., $g'(x) = f(x)$. (Note that this does *not* necessarily imply that we can compute g explicitly; take e.g. $f(x) = \mathrm{e}^{x^2}$. However, even in that case, we know that g *exists*.)

Problem 2 (15 points).
Find the derivative of the function

$$F(x) = \int_{\sqrt{x}}^{x} \frac{\mathrm{e}^t}{t}\, dt.$$

Solution.

We split up the integral into two integrals, and rewrite $F(x)$ as

$$F(x) = \int_{\sqrt{x}}^{0} \frac{\mathrm{e}^t}{t}\, dt + \int_{0}^{x} \frac{\mathrm{e}^t}{t}\, dt = -\int_{0}^{\sqrt{x}} \frac{\mathrm{e}^t}{t}\, dt + \int_{0}^{x} \frac{\mathrm{e}^t}{t}\, dt.$$

Since the upper limit of integration in the first integral is a function of x, we define $u = \sqrt{x}$; by the Fundamental Theorem of Calculus and the Chain Rule, we then have

$$F'(x) = \frac{d}{dx}\Big(-\int_{0}^{\sqrt{x}} \frac{\mathrm{e}^t}{t}\, dt + \int_{0}^{x} \frac{\mathrm{e}^t}{t}\, dt\Big) = -\frac{d}{du}\Big(\int_{0}^{u} \frac{\mathrm{e}^t}{t}\, dt\Big) \cdot \frac{du}{dx} + \frac{\mathrm{e}^x}{x}.$$

Since $\frac{du}{dx} = \frac{1}{2}\frac{1}{\sqrt{x}}$,

$$F'(x) = -\frac{\mathrm{e}^u}{u} \cdot \frac{1}{2}\frac{1}{\sqrt{x}} + \frac{\mathrm{e}^x}{x} = -\frac{\mathrm{e}^{\sqrt{x}}}{2x} + \frac{\mathrm{e}^x}{x}.$$

Problem 3 (15 points).

(a) Express $\displaystyle\int_1^2 (x^2 - 1)\, dx$ as a Riemann sum with n sample points. (Take the sample points to be the *right* end points.)

(b) Evaluate the sum in the limit as $n \to \infty$.

(Some useful identities:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2.\Big)$$

Solution.

(a) To set up the n-th Riemann sum, we divide the interval $[1, 2]$ into subintervals of width $\Delta x = \frac{2-1}{n} = \frac{1}{n}$. The end points of these intervals are given by $x_i = 1 + i \cdot \Delta x = 1 + \frac{i}{n}$. Hence, the Riemann sum corresponding to the choice of sample points $x_i^* = x_i$ is

$$R_n = \sum_{i=1}^{n} f(x_i) \cdot \Delta x = \sum_{i=1}^{n} (x_i^2 - 1) \cdot \Delta x = \sum_{i=1}^{n} \Big[\Big(1 + \frac{i}{n}\Big)^2 - 1\Big] \cdot \frac{1}{n}$$
$$= \sum_{i=1}^{n} \Big[1 + \frac{2i}{n} + \frac{i^2}{n^2} - 1\Big] \cdot \frac{1}{n} = \frac{1}{n^2} \cdot \sum_{i=1}^{n} \Big[2i + \frac{i^2}{n}\Big].$$

(b) To evaluate $\lim_{n\to\infty} R_n$, use the given identities to compute

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{1}{n^2}\cdot\left[2\sum_{i=1}^{n} i + \frac{1}{n}\sum_{i=1}^{n} i^2\right] = \lim_{n\to\infty}\frac{1}{n^2}\cdot\left[2\frac{n(n+1)}{2} + \frac{1}{n}\frac{n(n+1)(2n+1)}{6}\right]$$
$$= \lim_{n\to\infty}\frac{1}{n^2}\cdot\left[n^2+n+\frac{2n^2+3n+1}{6}\right] = \lim_{n\to\infty}\left[1+\frac{1}{n}+\frac{2}{6}+\frac{3}{6n}+\frac{1}{6n^2}\right] = 1+\frac{1}{3}=\frac{4}{3}.$$

Problem 4 (20 points).
Determine whether or not the following integral is convergent; if it is convergent, compute its value.

$$\int_0^2 z^2 \ln z\, dz$$

Solution.

First, note that the integral is *improper* of Type II, since the integrand is discontinuous at the lower limit (i.e., at 0): $\lim_{z\to 0^+} \ln z = -\infty$. Hence, to determine whether the integral is convergent, we have to check whether the limit

$$\lim_{t\to 0^+}\int_t^2 z^2\ln z\, dz$$

exists (is finite). Since for any $t>0$, the above integral is an ordinary definite integral, we can evaluate it using Integration by Parts (with $f(z)=\ln z$, $g'(z)=z^2$, and, hence, $f'(z)=\frac{1}{z}$, $g(z)=\frac{z^3}{3}$):

$$\lim_{t\to 0^+}\int_t^2 z^2\ln z\, dz = \lim_{t\to 0^+}\left[\ln z\cdot\frac{z^3}{3}\Big|_t^2 - \int_t^2 \frac{1}{z}\cdot\frac{z^3}{3}\, dz\right]$$
$$= \lim_{t\to 0^+}\left[\frac{8}{3}\ln 2 - \frac{t^3}{3}\ln t - \frac{1}{3}\cdot\frac{z^3}{3}\Big|_t^2\right] = \lim_{t\to 0^+}\left[\frac{8}{3}\ln 2 - \frac{8}{9} - \frac{t^3}{3}\ln t + \frac{t^3}{9}\right].$$

Now, $\lim_{t\to 0^+} t^3 = 0$; however, $\lim_{t\to 0^+} t^3\ln t = 0\cdot\infty$, which is an indeterminate form. Hence, we have to apply l'Hospital's Rule to find

$$\lim_{t\to 0^+} t^3\ln t = \lim_{t\to 0^+}\frac{\ln t}{\frac{1}{t^3}} = \frac{\infty}{\infty} = \lim_{t\to 0^+}\frac{\frac{1}{t}}{-\frac{3}{t^4}} = -\lim_{t\to 0^+}\frac{t^3}{3} = 0.$$

Therefore, the integral is convergent, and its value is $\frac{8}{3}\ln 2 - \frac{8}{9}$.

Problem 5 (15 points).
Evaluate the following integrals.

(a)

$$\int \frac{x+2}{\sqrt{x^2+4x}}\, dx$$

(b)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\theta}{1+\sin^2\theta}\, d\theta$$

Solution.

(a) We make the substitution $u = x^2 + 4x$, since the differential $du = (2x+4)dx = 2(x+2)dx$, which occurs in the integral (up to the factor 2). Hence,

$$\int \frac{x+2}{\sqrt{x^2+4x}}\,dx = \int \frac{x+2}{\sqrt{u}} \cdot \frac{du}{2(x+2)} = \int \frac{1}{2\sqrt{u}}\,du = \sqrt{u} = \sqrt{x^2+4x} + C.$$

(Another possibility is to substitute $u = \sqrt{x^2+4x}$.)

(b) Since the integrand is an even function and since the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is symmetric about 0, we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\theta}{1+\sin^2\theta}\,d\theta = 2\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{1+\sin^2\theta}\,d\theta.$$

We substitute $u = \sin\theta$, which implies $du = \cos\theta d\theta$. For $\theta = 0$, $u = \sin 0 = 0$, and for $\theta = \frac{\pi}{2}$, $u = \sin\frac{\pi}{2} = 1$. Hence, we have

$$2\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{1+\sin^2\theta}\,d\theta = 2\int_0^1 \frac{1}{1+u^2}\,du = 2\arctan(u)\Big|_0^1 = 2\Big(\frac{\pi}{4} - 0\Big) = \frac{\pi}{2}.$$

Problem 6 (20 points).
The solid S is obtained by rotating the region bounded by $y = \frac{x^2}{4}$ and $y = 5 - x^2$ about the x-axis.

(a) Sketch the region and the solid S.

(b) Sketch a typical disk, or washer (whichever might be required), and find the volume of S.

Solution (up to sketches).

(a) For the points of intersection of the two curves, we find $\frac{x^2}{4} = 5 - x^2$ and hence $x^2 = 4$ or $x = \pm 2$. The corresponding y-value is $5 - (\pm 2)^2 = 1$; therefore, the points are given by $(-2, 1)$ and $(2, 1)$. We are concerned with the region enclosed by the two curves, i.e., with $\frac{x^2}{4} \le y \le 5 - x^2$ for $-2 \le x \le 2$. The solid S is obtained by rotating that region about the x-axis.

(b) The cross-section of S with any plane P_x through $x \in [-2, 2]$ is a washer with outer radius $5 - x^2$ and inner radius $\frac{x^2}{4}$. The area of this washer is given by

$$\begin{aligned} A(x) = \pi[(\text{outer radius})^2 - (\text{inner radius})^2] &= \pi\Big[(5-x^2)^2 - \Big(\frac{x^2}{4}\Big)^2\Big] \\ &= \pi\Big[25 - 10x^2 + \frac{15}{16}x^4\Big]. \end{aligned}$$

Therefore, the volume of S is

$$\begin{aligned} V = \int_{-2}^{2} A(x)\,dx = \pi\int_{-2}^{2} \underbrace{\Big[25 - 10x^2 + \frac{15}{16}x^4\Big]}_{\text{even}}\,dx &= 2\pi\int_0^2 \Big[25 - 10x^2 + \frac{15}{16}x^4\Big]\,dx \\ &= 2\pi\Big[25x - \frac{10}{3}x^3 + \frac{3}{16}x^5\Big]\Big|_0^2 = \frac{176}{3}\pi. \end{aligned}$$