Test Total

Name KEY

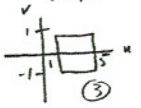
3450:223 Calculus III Final Exam - 150 pts.
Clemons

Spring '02

1. Use a transformation to evaluate $\iint_R \frac{e^{y+3x}}{y-2x}\,dA$, where R is the rectangle enclosed by the lines $y = 2x+1$, $y = 2x+5$, $y = 1-3x$ and $y = -1-3x$.

Let $u = y-2x$, $v = y+3x$

Then $u=1, u=5$
$v=1, v=-1$



3

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix}} = -\frac{1}{5} \quad 4$$

$$\iint_S \frac{e^v}{u}\left|-\frac{1}{5}\right| dA = \int_{-1}^{1}\int_1^5 \frac{1}{5}\frac{e^v}{u}\,dv \quad -1$$

$$= \frac{1}{5}\ln 5 \int_{-1}^{1} e^v\,dv$$

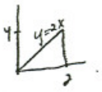
$$= \frac{\ln 5}{5}\left(e - \frac{1}{e}\right) \quad 4$$

12 pts

2. Find the surface area of the portion of the surface $z = 4y + 3x^2$ that lies between $y = 2x$, $y = 0$ and $x = 2$.

$z = f(x,y) = 3x^2 + 4y \qquad f_x = 6x, \; f_y = 4$

$$SA = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dA \quad 3$$



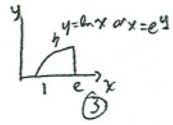
$$= \iint_R \sqrt{36x^2+17}\, dA \quad 3$$

$$= \int_0^2 \int_0^{2x} \sqrt{36x^2+17}\, dy\, dx = \int_0^2 2x\sqrt{36x^2+17}\, dx \quad 2$$

$$= \frac{1}{36}\cdot\frac{2}{3}(36x^2+17)^{3/2}\Big|_0^2 = 36.53 \quad 3$$

11 pts

3. Evaluate the double integral $\int_1^e \int_0^{\ln x} y\,dy\,dx$, by interchanging the order of integration.



3

$$= \int_0^1 \int_{e^y}^{e} y\,dx\,dy = \int_0^1 (e - e^y)\,y\,dy \quad 3 \quad 3$$

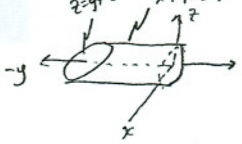
$$= \frac{e}{2} - (ye^y - e^y)\Big|_0^1 = \frac{e}{2} - (e - e - (0-1))$$

$$= \frac{e}{2} - 1 \quad 2$$

11 pts

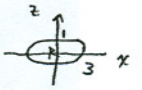
4. Let D be the solid enclosed by $z = y+3$, $x^2 + 9z^2 = 9$ and $y = 0$.

(a) Sketch the solid region D.



4 pts

(b) Set-up [**do not solve**] the triple integral in Cartesian coordinates necessary to find the volume of the region D.



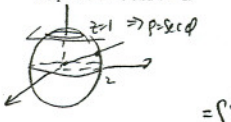
$$\iint_R \int_0^{z-3} dy\, dA$$

$$= \int_{-1}^{1} \int_{-\sqrt{9-9z^2}}^{\sqrt{9-9z^2}} \int_0^{z-3} dy\,dx\,dz \quad 3 \quad 2 \quad 2 \quad 1$$

$$\text{or} = \int_{-3}^{3} \int_{-\sqrt{1-\frac{1}{9}x^2}}^{\sqrt{1-\frac{1}{9}x^2}} \int_0^{z-3} dy\,dz\,dx$$

7 pts

5. Using spherical coordinates, find the volume of the solid region E bounded by the sphere $x^2+y^2+z^2 = 4$ and the planes $z = 0$ and $z = 1$.



when $z=1$ & $x=0$
$y^2 = 4-1 = 3$
$y = \sqrt{3}$

$\tan\varphi = \frac{\sqrt{3}}{1}$
$\varphi = \frac{\pi}{3}$

$z = 1$
$\Rightarrow \rho = \sec\phi$

$$V = \iiint_{E_1} 1\,dv + \iiint_{E_2} 1\,dv$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\pi/2}\int_0^{1} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta + \int_0^{2\pi}\int_0^{\pi/3}\int_0^{\sec\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \frac{2\pi}{3}\int_{\frac{\pi}{3}}^{\pi/2} \sin\varphi\, d\varphi + 2\pi \int_0^{\pi/3}\int_0^{\sec\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi$$

$$= \frac{2\pi}{3}\left(\cos\frac{\pi}{3} - \cos\frac{\pi}{2}\right) + \frac{2\pi}{3}\int_0^{\pi/3} \sec^3\varphi \sin\varphi\, d\varphi$$

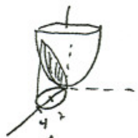
$$= \frac{\pi}{3} + \frac{2\pi}{3}\int_0^{\pi/3} \tan\varphi \sec^2\varphi\, d\varphi$$

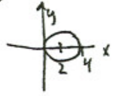
$$= \frac{\pi}{3} + \frac{2\pi}{3}\cdot\frac{1}{2}\tan^2\varphi \Big|_0^{\pi/3}$$

$$= \frac{\pi}{3} + \frac{\pi}{3}\sqrt{3}$$

11 pts

6. Using cylindrical coordinates, evaluate $\iiint_E \frac{y}{x}\,dV$, where E is the region region bounded between $z = x^2+y^2$ and $z = 0$ and inside $(x-2)^2 + y^2 = 4$.





$r = 4\cos\theta$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\iint_R \int_0^{x^2+y^2} \frac{y}{x}\, dz\, dA$$

$$= \iint_R \frac{y}{x}(x^2+y^2)\, dA \quad 3$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2}\int_0^{4\cos\theta} r^2 \frac{\sin\theta}{\cos\theta}\, r\,dr\,d\theta \quad 4 \quad 2$$

$$= 64\int_{-\frac{\pi}{2}}^{\pi/2} \cos^3\theta \sin\theta\, d\theta$$

$$= -\frac{64}{4}\cos^4\theta \Big|_{-\frac{\pi}{2}}^{\pi/2} = 0$$

11 pts

7. Find the volume of the tetrahedron bounded by the coordinate planes and the plane with points $(3,2,0)$, $(2,0,2)$ and $(0,0,3)$.

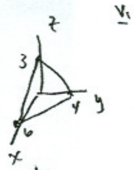
$(3,2,0), (2,0,2), (0,0,3)$

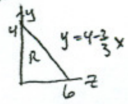
$\underline{v}_1 = \langle 3,2,-3\rangle, \; \underline{v}_2 = \langle 2,0,-1\rangle$

$$\underline{n} = \begin{vmatrix} i & j & k \\ 3 & 2 & -3 \\ 2 & 0 & -1 \end{vmatrix} = -2i - 3j - 4k$$

$\underline{n}\cdot\langle x, y, z-3\rangle = 0 \Rightarrow 2x+3y+4z = 12$

$\Rightarrow z = 3 - \frac{1}{2}x - \frac{3}{4}y$





$$\iint_R \left(3 - \frac{1}{2}x - \frac{3}{4}y\right) dA = \int_0^6 \int_0^{4-\frac{2}{3}x} \left(3 - \frac{1}{2}x - \frac{3}{4}y\right) dy\,dx \quad 6$$

$$= \int_0^6 \left(3 - \frac{1}{2}x\right)\left(4 - \frac{2}{3}x\right) - \frac{3}{8}\left(4 - \frac{2}{3}x\right)^2 dx \quad 4$$

$$= \int_0^6 12 - 2x - 2x + \frac{1}{3}x^2 - \frac{3}{8}\left(16 - \frac{16}{3}x + \frac{4}{9}x^2\right) dx$$

$$= \int_0^6 \frac{1}{6}x^2 - 2x + 6\,dx$$

$$= \frac{1}{18}x^3 - x^2 + 6x \Big|_0^6 = 12 \quad 5$$

15 pts

8. Set-up and evaluate using Green's Theorem a line integral for the Total Work done in moving a point through the Force field $\vec{F} = (x^2y)\vec{i} + x\vec{j}$ around the triangular region bounded by the x-axis, $x = 1$ and the line $y = 2x$, in a counter-clockwise fashion.

$M = x^2y \qquad N = x$

$N_x - M_y = 1 - x^2$

$$= \int_C (x^2y)\,dx + x\,dy$$

$$= \iint_R (1-x^2)\,dA = \int_0^1 \int_0^{2x} (1-x^2)\,dy\,dx$$

$$= \int_0^1 2x - 2x^3\,dx = x^2 - \frac{2}{4}x^4 \Big|_0^1 = \frac{1}{2}$$

10 pts

9. Use a suitable *potential function* to evaluate the line integral
$\int_C (3x+y+1)\,dx + (x+4y+2)\,dy$ where C is any curve connecting $(-1,2)$ to $(0,1)$.

$f_x = 3x+y+1 \quad f_y = x+4y+2$

(2) $f = \frac{3}{2}x^2 + yx + x + g(y)$

(2) $f_y = x + g' = x+4y+2$

$g' = 4y+2$

(2) $g = 2y^2 + 2y$

$f = \frac{3}{2}x^2 + yx + x + 2y^2 + 2y$

$= f(0,1) - f(-1,0)$

$= 4 - \left(\frac{3}{2} - 2 - 1 + 8 + 4\right)$

$4 = -\frac{13}{2}$

10 pts

10. Given that $(0,0)$, $(-1,1/2)$ and $(-2,1)$ are critical points of $f(x,y) = x^3 - 2y^2 - 2y^4 + 3x^2y$, use the second derivative test to classify them.

$f_x = 3x^2 + 6xy \quad f_y = -4y - 8y^3 + 3x^2$

$f_{xx} = 6x + 6y \quad f_{yy} = -4 - 24y^2$

$f_{xy} = 6x$

$D = \begin{vmatrix} 6x+6y & 6x \\ 6x & -4-24y^2 \end{vmatrix} = -\left(6(x+y)\cdot 4(1+6y^2) + 36x^2\right)$ (4)

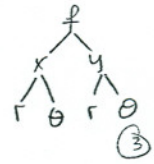
@ $(0,0)$ $\quad D(0,0) = 0$ $\quad$ test fails

@ $(-1,\frac{1}{2})$ $\quad D(-1,\frac{1}{2}) = -6$ $\quad$ Saddle

@ $(-2,1)$ $\quad D(-2,1) = 24$, $f_{xx}(-2,1) \le 0$ $\quad$ local max

10 pts

11. Given $f(x,y)$ is a differentiable function with $x = r\cos\theta$ and $y = r\sin\theta$, show that $f_\theta = -f_x r\sin\theta + f_y r\cos\theta$.



(3)

$f_\theta = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta}$ (3)

$= f_x(-r\sin\theta) + f_y(r\cos\theta)$. (3)

9 pts

12. For the space curve $\vec{r}(t) = \langle e^t, e^{(-t)}, \sqrt{2}t \rangle$:
Calculate the arclength for $0 \le t \le 1$.

$\int_0^1 \sqrt{x'^2 + y'^2 + z'^2}\,dt$ (2) $= \int_0^1 \sqrt{(e^t)^2 + (-e^{-t})^2 + (\sqrt{2})^2}\,dt$ (2)

$= \int_0^1 \sqrt{e^{2t} + \frac{1}{e^{2t}} + 2}\,dt = \int_0^1 \sqrt{\frac{e^{4t} + 2e^{2t} + 1}{e^{2t}}}\,dt$

$= \int_0^1 \frac{1}{e^t}\sqrt{(e^{2t}+1)^2}\,dt = \int_0^1 e^t + e^{-t}\,dt$ (2)

$= 2\int_0^1 \cosh t\,dt = 2\sinh t\,\Big|_0^1 = 2\sinh(1)$

(2)

12 pts

13. Given $f(x,y) = e^{-x}\sec y$:
(a) Find the tangent plane at the point $P(0,\pi/4)$.

(3)

$\langle f_x(0,\frac{\pi}{4}), f_y(0,\frac{\pi}{4}), -1 \rangle \cdot \langle x-0, y-0, z-f(0,\pi/4) \rangle = 0$

$f_x = -e^{-x}\sec y$, $f_x(0,\pi/4) = -\sqrt{2}$

$f_y = e^{-x}\sec y\tan y\Big|_0^{\pi/4} = \sqrt{2}$

$\langle -\sqrt{2}, \sqrt{2}, -1 \rangle \cdot \langle x, y-\pi/4, z-\sqrt{2} \rangle = 0$ (3)

6 pts

(b) Use the linear approximation to approximate $f(.01, \pi/4 - .01)$.

$\langle -\sqrt{2}, \sqrt{2}, -1 \rangle \cdot \langle .01, \frac{\pi}{4} - .01 - \frac{\pi}{4}, z - \sqrt{2} \rangle = 0$

(3)

$-\sqrt{2}(.01) + \sqrt{2}(-.01) - z + \sqrt{2} = 0$

$z = \sqrt{2}(1 - .02)$ (2)

5 pts

(c) Calculate the directional derivative of $f(x,y)$ at $P(0,\pi/4)$ toward the direction of the origin.

$\nabla f(0,4) = \langle -\sqrt{2}, \sqrt{2} \rangle$

$\underline{u} = \frac{(0,0) - (0,\frac{\pi}{4})}{\|(0,0) - (0,\frac{\pi}{4})\|} = \langle 0, -1 \rangle$ (3)

$D_u f = \langle -\sqrt{2}, \sqrt{2} \rangle \cdot \langle 0, -1 \rangle = -\sqrt{2}$

6 pts