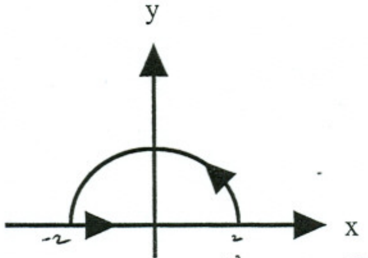
150 Points

Calc III - 3450:223 NAME Key

FINAL Fall97 ROW

Show **ALL** your work.

1. Evaluate $\int_C (x+y)\,ds$ along the portions of the curves $x^2+y^2=4$ and $y=0$ as shown in the figure.



13 Points

$\rightarrow$ $x = t$, $y = 0$ (2) $\quad \int_{-2}^{2} (t+0)\sqrt{1^2+0^2}\,dt$ (2) $= \frac{t^2}{2}\Big|_{-2}^{2} = 2-(2) = 0$ (2)

$\curvearrowleft$ $x = 2\cos t$, $y = 2\sin t$ (2)

$\int_0^{\pi} (2\cos t + 2\sin t)\sqrt{4\sin^2 t + 4\cos^2 t}\,dt$ (3)

$= 4\int_0^{\pi} (\cos t + \sin t)\,dt = 4[\sin t - \cos t]\Big|_0^{\pi}$

$= 4[1 - -1] = 8$ (2)

2. Find the curl and divergence of $\mathbf{F} = xe^y\mathbf{i} - ze^{-y}\mathbf{j} + y\ln z\,\mathbf{k}$.

12 Points

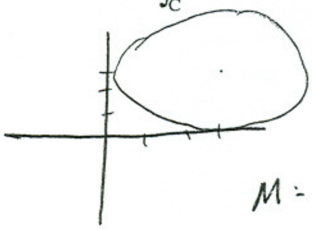
$\nabla\cdot\vec{F} = e^y + ze^{-y} + \frac{y}{z}$ (6)

$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xe^y & -ze^{-y} & y\ln z \end{vmatrix} = (\ln z + e^{-y})\hat{i} - \hat{j}(0-0) + \hat{k}(0 - xe^y)$ (6)

$= (\ln z + e^{-y})\hat{i} - xe^y\hat{k}$

25 Points

3. Evaluate $\int_C (7x - 5y)dx + (6x - 7y)dy$ where C is the circle $(x - 3)^2 + (y - 3)^2 = 9$.



Try Green's Thm

12 Points

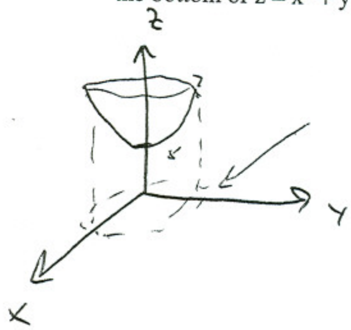
$M = 7x - 5y \qquad N = 6x - 7y$

$M_y = -5 \qquad N_x = 6 \quad 4$

$\int\int (6 - -5)\, dA = 11 \cdot \iint_{O} dA = 11 \cdot \pi (3)^2 = 99\pi$

4 $\qquad$ 4

4. Find the flux of the vector field $\mathbf{F} = 8x\mathbf{i} + 8y\mathbf{j} + 4\mathbf{k}$ outward through the surface cut from the bottom of $z = x^2 + y^2 + 5$ by $z = 7$.



13 Points

R $\quad x^2 + y^2 = 2$

Let $H = x^2 + y^2 + 5 - z$

$\nabla H = \langle 2x, 2y, -1 \rangle$

2 $\qquad$ 2 $\qquad$ 2

$\text{Flux} = \iint_S \vec{F} \cdot \hat{n}\, dS = \iint_{O} \dfrac{\langle 8x, 8y, 4 \rangle \cdot \langle 2x, 2y, -1 \rangle}{\cancel{\sqrt{4x^2 + 4y^2 + 1}}} \cdot \dfrac{\cancel{\sqrt{4x^2 + 4y^2 + 1}}}{|\langle 2x, 2y, -1 \rangle \cdot \langle 0, 0, 1 \rangle|}\, dA$

2

$= \int_0^{2\pi} \int_0^{\sqrt{2}} (16x^2 + 16y^2 - 4)\, r\, dr\, d\theta$ $\quad$ 2

$= \int_0^{2\pi} \int_0^{\sqrt{2}} (16r^3 - 4r)\, dr\, d\theta$

$= 2\pi (4r^4 - 2r^2) \Big|_0^{\sqrt{2}}$ $\quad$ 3

$= 2\pi \cdot [16 - 4] = 24\pi$

25 Points

5. Let $\mathbf{F} = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$. Is $\mathbf{F}$ conservative? Also evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is given by $\mathbf{r}(t) = (e^t \sin t)\,\mathbf{i} + (e^t \cos t)\,\mathbf{j}$, $0 \le t \le \pi$.

12 Points

$P = 3 + 2xy \qquad Q = x^2 - 3y^2$

$P_y = 2x \qquad Q_x = 2x \qquad$ so conservative (4)

Find f so $\nabla f = f_x \hat{i} + f_y \hat{j} = (3+2xy)\hat{i} + (x^2 - 3y^2)\hat{j}$

$f_x = 3 + 2xy$

$f = 3x + x^2 y + g(y)$

$f_y = \cancel{2xy}\, x^2 + g'(y) = x^2 - 3y^2$

$g'(y) = -3y^2$

so $g(y) = -y^3 + c$

$\Rightarrow f = 3x + x^2 y - y^3 + c$ (4)

$\int \vec{F}\cdot d\vec{r} = f(0, -e^{\pi}) - f(0,1)$

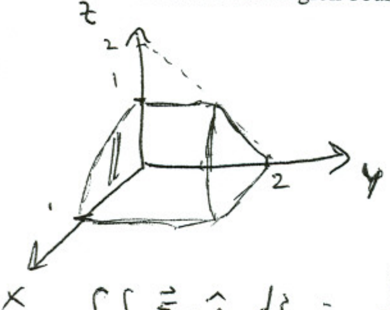
$\vec{r}(0) = \langle 0, 1 \rangle$

$\vec{r}(\pi) = \langle 0, -e^{\pi} \rangle$

$= e^{3\pi} + 1$ (4)

6. Calculate the flux of $\mathbf{F} = \frac{1}{2}x^2\mathbf{i} + \sin(xz)\,\mathbf{j} + (xz + e^{x^2 y})\mathbf{k}$ across the surface S where S is the surface of the region bounded by $x = 0$, $y = 0$, $z = 0$, $z = 1 - x^2$ and $y + z = 2$.

13 Points



$\iint \vec{F}\cdot\hat{n}\, dS = \iiint \nabla\cdot\vec{F}\, dV$ by the div thm

(6) $= \int_0^1 \int_0^{1-x^2} \int_0^{2-z} (x + 0 + x)\, dy\, dz\, dx$ (3)

$= \int_0^1 \int_0^{1-x^2} 2x(2-z)\, dz\, dx = \int_0^1 2x\left(2z - \frac{z^2}{2}\right)\Big|_0^{1-x^2} dx$

$= \int_0^1 2x\left[2 - 2x^2 - \frac{1}{2}(1 - 2x^2 + x^4)\right] dx$

$= \int_0^1 [3x - 2x^3 - x^5]\, dx$ (4)

$= \frac{3x^2}{2} - \frac{2x^4}{4} - \frac{x^6}{6}\Big|_0^1 = \frac{3}{2} - \frac{1}{2} - \frac{1}{6} = \frac{5}{6}$

25 Points

7. Find the vector componopts of $2\mathbf{i} + 3\mathbf{j}$ which are parallel and perpendicular to $-\mathbf{i} + 2\mathbf{j}$.

10 Points

Parallel $\left(\frac{\langle 2,3\rangle \cdot \langle -1,2\rangle}{\sqrt{1+4}}\right)\frac{\langle -1,2\rangle}{\sqrt{1+4}}$ 2 2

$= \frac{4}{5}\langle -1, 2\rangle = \langle -\frac{4}{5}, \frac{8}{5}\rangle$ 1

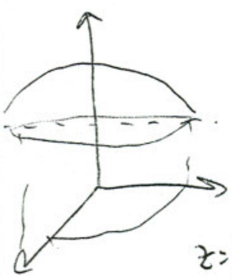
Perpendicular $\langle 2,3\rangle - \frac{4}{5}\langle -1,2\rangle = \langle 2+\frac{4}{5}, 3-\frac{8}{5}\rangle$

5 $= \langle 14/5, 7/5\rangle$

8. Let V be the solid that lies above $z = 7$ and inside $x^2 + y^2 + z^2 = 64$. Let the density of this solid be given by xyz. SET UP BUT DO NOT EVALUATE INTEGRALS

a) in spherical coordinates to find the mass of V.

8 Points



$$\text{Mass} = \int_0^{2\pi}\int_0^{\cos^{-1}\frac{7}{8}}\int_{\frac{7}{\cos\phi}}^{8} (\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)\rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

1 2 2 1 2

$z = 7 = \rho\cos\phi \Rightarrow \rho = \frac{7}{\cos\phi}$

at intersection $\rho = 8 \Rightarrow \frac{7}{8} = \cos\phi$

$\phi = \cos^{-1} 7/8$

b) in cylindrical coordinates to find the volume of V.

7 Points

$$\text{Volume} = \int_0^{2\pi}\int_0^{\sqrt{15}}\int_7^{\sqrt{64-r^2}} r\, dz\, dr\, d\theta$$

1 2 2 2

$x^2 + y^2 + 49 = 64$

$x^2 + y^2 = 64 - 49 = 15$

25 Points

9. The point P(2, 1, −1) lies on both $F(x, y, z) = -x + y^2 + z^2 = 0$ and $G(x, y, z) = x^2 + 2y^2 + 3z^2 - 9 = 0$. Find an equation of the line through P that is tangent to both surfaces F and G. Hint: The line is perpendicular to both normals at P.

$$x = 2 + t(-4)$$
$$y = 1 + t(-14)$$
$$z = -1 + t(-12)$$

4

13 Points

$\vec{n}$ to F : $\nabla F = \langle -1, 2y, 2z \rangle = \langle -1, 2, -2 \rangle$ 2

$\vec{n}$ to G : $\nabla G = \langle 2x, 4y, 6z \rangle = \langle 4, 4, -6 \rangle$ 2

direction : $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & -2 \\ 4 & 4 & -6 \end{vmatrix} = (-12+8)\hat{i} - \hat{j}(6+8) + \hat{k}(-4-8)$

$= -4\hat{i} - 14\hat{j} - 12\hat{k}$

5

10. Let w = f(x, y) where x = u − 2v and y = u + 2v.

a) Find $\frac{\partial w}{\partial u}$ in terms of f_x and f_y.

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$$

$$= f_x \cdot 1 + f_y \cdot 1$$

2 2

4 Points

b) Find $\frac{\partial^2 w}{\partial v \partial u}$ in terms of f_{xx}, f_{yy}, and f_{xy}.

$$\frac{\partial}{\partial v}\left(\frac{\partial w}{\partial u}\right) = \frac{\partial}{\partial v}\left[f_x + f_y\right]$$

8 Points

$$= \left[f_{xx}\frac{\partial x}{\partial v} + f_{xy}\frac{\partial y}{\partial v} + f_{yx}\frac{\partial x}{\partial v} + f_{yy}\frac{\partial y}{\partial v}\right]$$

2 2 2 2

$$= \left[-2f_{xx} + 2f_{xy} - 2f_{yx} + 2f_{yy}\right]$$

25 Points

$$= -2f_{xx} + 2f_{yy}$$

11. Find the rate of change of $f(x, y) = e^{x^2y} - 3xy$ at the point (2, 3) in the direction of $-\mathbf{i} + 2\mathbf{j}$.

12 Points

$D_{\vec{u}} f = \nabla f \cdot \vec{u}$

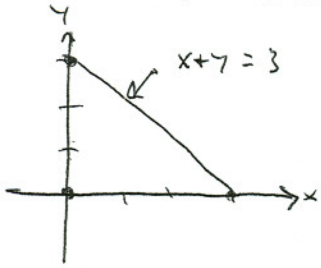
$= \left\langle 2xye^{x^2y} - 3y,\ x^2e^{x^2y} - 3x \right\rangle_{(2,3)} \cdot \dfrac{\langle -1, 2\rangle}{\sqrt{5}}$

5 5

$= \langle 12e^{12} - 9,\ 4e^{12} - 6 \rangle \cdot \dfrac{\langle -1, 2 \rangle}{\sqrt{5}}$

$= \dfrac{-12e^{12} + 9 + 8e^{12} - 12}{\sqrt{5}} = \dfrac{-4e^{12} - 3}{\sqrt{5}}$ 2

12. Find the absolute maximum and minimum values obtained by the function $f(x, y) = 2xy - 2x - 2y + 1$ on the triangular region R in the xy–plane with vertices at the points (0, 0), (0, 3), and (3, 0).



13 Points

Check inside

$f_x = 2y - 2 = 0 \Rightarrow y = 1$

$f_y = 2x - 2 = 0 \Rightarrow x = 1$ 4

$f(1,1) = 2 - 2 - 2 + 1 = -1$

$y = 0,\ 0 \le x \le 3$

$(x,0) = -2x + 1$ $f(0,0) = 1$ 2

$f(3,0) = -5$

Max = 1 at (0,0)

min: −5 at (3,0) 1
(0,3)

$= 0 \quad 0 \le y \le 3$

$(0, y) = -2y + 1$ $f(0,0) = 1$ 2

$f(0,3) = -5$

$y = 3 - x$

$(x, 3-x) = 2x(3-x) - 2x - 2(3-x) + 1 = 6x - 2x^2 - \cancel{2x} - 6 + \cancel{2x} + 1$

$= -2x^2 + 6x - 5$

$'(x, 3-x) = -4x + 6 = 0 \Rightarrow x = 3/2$ 4

$\Rightarrow y = 3/2$

$(3/2, 3/2) = 2 \cdot \frac{9}{4} - 3 - 3 + 1 = -5 + \frac{9}{2} = -1/2$

25 Points