Calc III - 3450:223
EXAM # 1 Fall97

NAME Key
ROW

100 Points

Show **ALL** your work.

1. Given vectors **a**, **b**, and **c**, show using magnitudes and/or the dot product and/or the cross product how you would

a) decide if **a** and **b** are parallel

$\vec{a} \times \vec{b} = \vec{0} \Rightarrow$ parallel

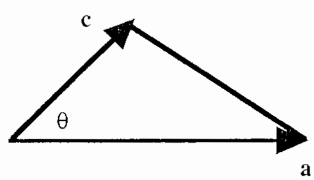
2 Points

b) decide if **a** and **b** are orthogonal

$\vec{a} \cdot \vec{b} = 0 \Rightarrow$ orthogonal

2 Points

c) calculate the area of the triangle determined by **a** and **c**



$A = \frac{1}{2}|\vec{a} \times \vec{c}|$

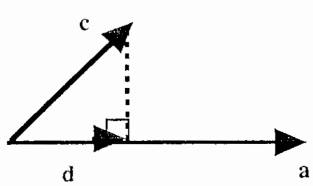
2 Points

d) calculate the angle θ

$\cos\theta = \dfrac{\vec{a} \cdot \vec{c}}{|\vec{a}||\vec{c}|}$

2 Points

e) find the vector **d**

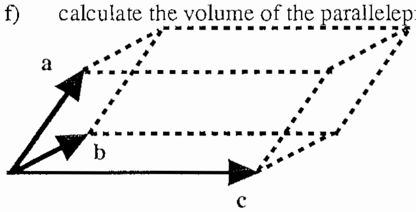


$\vec{d} = \left(\vec{c} \cdot \frac{\vec{a}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|}$

(2) (1)

3 Points

f) calculate the volume of the parallelepiped determined by **a**, **b**, and **c**.



$|\vec{a} \cdot (\vec{c} \times \vec{b})|$

3 Points

2. Identify the given surface and convert its equation to cylindrical coordinates: $x^2 + y^2 = 2z$

paraboloid (4)

8 Points

$r^2 = 2z$ (4)

22 Points

3. The position of a projectile is given by $\mathbf{r}(t) = 3\sin(t)\,\mathbf{i} + 4t\,\mathbf{j} + 3\cos(t)\,\mathbf{k}$. Find the following:

a) The projectile's velocity, $\mathbf{v}$

$$\vec{v} = \vec{r}\,'(t) = 3\cos t\,\hat{i} + 4\hat{j} - 3\sin t\,\hat{k}$$

2 Points

b) The speed of the projectile

$$\text{Speed} = \sqrt{9\cos^2 t + 16 + 9\sin^2 t} = \sqrt{25} = 5$$

$$= |\vec{r}\,'|$$

2 Points

c) The unit tangent vector, $\mathbf{T}$

$$\vec{T} = \frac{\vec{r}\,'}{|\vec{r}\,'|} = \frac{3\cos t\,\hat{i} + 4\hat{j} - 3\sin t\,\hat{k}}{5}$$

2 Points

d) The unit normal vector, $\mathbf{N}$

$$\vec{N} = \frac{\vec{T}\,'}{|\vec{T}\,'|} \quad (1) \qquad \vec{T}\,' = \frac{1}{5}\left[-3\sin t\,\hat{i} + 0\hat{j} - 3\cos t\,\hat{k}\right] \quad (1)$$

$$|\vec{T}\,'| = \frac{1}{5}\sqrt{9\sin^2 t + 9\cos^2 t} = \frac{3}{5} \quad (1)$$

$$\vec{N} = -\sin t\,\hat{i} - \cos t\,\hat{k}$$

3 Points

e) The curvature, κ

$$\kappa = \frac{|\vec{T}\,'|}{|\vec{r}\,'|} = \frac{3/5}{5} = 3/25$$

(1)

3 Points

f) The acceleration of the projectile written in terms of $\mathbf{T}$ and $\mathbf{N}$. You do not have to write out $\mathbf{T}$ and $\mathbf{N}$ in your answer.

$$\vec{r}\,'' = \vec{a} = -3\sin t\,\hat{i} + 0\hat{j} - 3\cos t\,\hat{k} = a_T\vec{T} + a_N\vec{N}$$

$$a_T = \vec{a}\cdot\vec{T} = \langle -3\sin t, 0, -3\cos t\rangle \cdot \tfrac{1}{5}\langle 3\cos t\,\hat{i} + 4\hat{j} - 3\sin t\,\hat{k}\rangle$$

$$= -9\sin t\cos t + 9\cos t\sin t = 0 \quad (3)$$

6 Points

$$a_N = \vec{a}\cdot\vec{N} = \langle -3\sin t, 0, -3\cos t\rangle \cdot \langle -\sin t, 0, -\cos t\rangle$$

$$= 3\sin^2 t + 3\cos^2 t = 3 \quad (3)$$

18 Points

4. Find an equation for the plane consisting of all points that are equidistant from the points $(-4, 2, 1)$ and $(2, -4, 3)$.

A B

10 Points

$$pt : \frac{1}{2}\left[(-4,2,1)+(2,-4,3)\right] = \frac{1}{2}(-2,-2,4)$$
$$= (-1,-1,2) \quad (3)$$

$$\hat{n} = \vec{AB} = \langle 6,-6,2\rangle \quad (3)$$

$$\text{plane} = \langle x+1, y+1, z-2\rangle \cdot \langle 6,-6,2\rangle = 0$$
(4)

5. Find AC×AD and AB·(AC×AD) given the points A(1,0,1), B(2,3,0), C(−1,1,4), and D(0,3,2).

10 Points

$$AC = \langle -2, 1, 3\rangle \qquad AD = \langle -1,3,1\rangle$$
$$AB = \langle 1,3,-1\rangle \quad (3)$$

$$AC\times AD = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 3 \\ -1 & 3 & 1 \end{vmatrix} = -8\hat{i} - \hat{j} - 5\hat{k} \quad (4)$$

$$\vec{AB}\cdot\langle -8,-1,-5\rangle = \langle 1,3,-1\rangle \cdot \langle -8,-1,-5\rangle$$
$$= -8 - 3 + 5 = -6 \quad (3)$$

6. Find parametric equations for the line tangent to $\mathbf{r}(t)$ at $t = 1$ if $\mathbf{r}'(t) = 6t\,\mathbf{i} - 2e^{2t}\,\mathbf{j} + \frac{1}{1+t}\,\mathbf{k}$ and $\mathbf{r}(0) = 2\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}$.

10 Points

$$\vec{r}\,' = 6t\,\hat{i} - 2e^{2t}\hat{j} + \frac{1}{1+t}\hat{k}$$
$$\vec{r} = 3t^2\,\hat{i} - e^{2t}\hat{j} + \ln|1+t|\,\hat{k} + \vec{C} \quad (4)$$

$$\vec{r}(0) = \langle 2,-1,4\rangle = \langle 0,-1,0\rangle + \vec{C} \Rightarrow \vec{C} = \langle 2,0,4\rangle$$
$$\vec{r} = (3t^2+2)\,\hat{i} - e^{2t}\hat{j} + \left[\ln|1+t|+4\right]\hat{k} \quad (2)$$

$$\vec{r}\,'(1) = \langle 6, -2e^2, \tfrac{1}{2}\rangle \qquad x = 5 + t(6)$$
$$\vec{r}(1) = \langle 5, -e^2, \ln 2 + 4\rangle \qquad y = -e^2 + t(-2e^2)$$
(2)
$$z = \ln 2 + 4 + t(1/2)$$
(2)

30 Points

7. Convert the following equation in spherical coordinates to one in rectangular coordinates and sketch its graph: $\rho^2[\sin^2\phi \sin^2\theta + \cos^2\phi] = 9$.

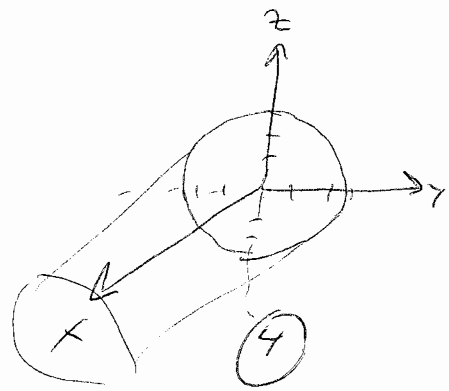
10 Points

$\rho\cos\phi = z$ $\quad \rho\sin\phi\sin\theta = r\sin\theta = y$

(3) (3)

$y^2 + z^2 = 9$

cylinder



(4)

8. Find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.

10 Points

$10x + 2y - 2z - 5 = 0$

pt on $5x + y - z = 1$ is $(0, 1, 0)$ (3)

(4)

$$d = \frac{|10\cdot 0 + 2\cdot 1 - 2\cdot 0 - 5|}{\sqrt{100 + 4 + 4}} = \frac{3}{\sqrt{108}} = \frac{3}{6\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

(2)

$$= \frac{\sqrt{3}}{6}$$

(1)

9. A projectile is fired horizontally with initial speed of 500 m/s from a position 200 m above the ground. Determine the speed of the projectile when it hits the ground.

10 Points

$x = 500t$ (1)

$\vec{r} = x\hat{i} + y\hat{j}$

$y = -\frac{1}{2}(9.8)t^2 + 200$ (2)

Hits ground $\Rightarrow$ $y = 0 = -\frac{1}{2}(9.8)t^2 + 200$ $\Rightarrow$ $t = \sqrt{\frac{200 \cdot 2}{9.8}}$

(4)

Speed $= |\vec{r}\,'| = |500\hat{i} + -9.8t\hat{j}|$

$$= \sqrt{(500)^2 + (9.8)^2 \cdot \frac{200 \cdot 2}{9.8}} \simeq 504 \text{ m/s}$$

(3)

30 Points