CALCULUS III - 3450:223
TEST # 1

NAME Key

100 Points

1. Write **i** + **j** + **k** as the sum of a vector parallel to and orthogonal to 4**i** + 2**j**.

10 Points

$$\text{Component along } 4\hat{i}+2\hat{j} = \frac{(\hat{i}+\hat{j}+\hat{k})\cdot(4\hat{i}+2\hat{j})}{\sqrt{16+4}} = \frac{6}{\sqrt{20}}$$

2 2

$$\text{projection} = \frac{6}{\sqrt{20}}\cdot\frac{(4\hat{i}+2\hat{j})}{\sqrt{20}} = \frac{12}{20}\cdot(2\hat{i}+\hat{j}) = \frac{3}{5}(2\hat{i}+\hat{j})$$

3

$$\text{orthogonal} = (\hat{i}+\hat{j}+\hat{k}) - \frac{3}{5}(2\hat{i}+\hat{j}) = -\frac{1}{5}\hat{i} + \frac{2}{5}\hat{j} + \hat{k}$$

3

$$(\hat{i}+\hat{j}+\hat{k}) = \frac{3}{5}(2\hat{i}+\hat{j}) + \frac{1}{5}(-\hat{i}+2\hat{j}+5\hat{k})$$

$$\left(\frac{6}{5}\hat{i}+\frac{3}{5}\hat{j}\right) + \left(\frac{-1}{5}\hat{i}+\frac{2}{5}\hat{j}+1\hat{k}\right)$$

Sample Test
Quiz #2
pg 737
13, 15

2. Given the vectors **a**, **b**, and **c**, show, using vector operations, how you would

a) decide if **a** and **b** are orthogonal

Class Notes

If $\vec{a}\cdot\vec{b}=0 \Rightarrow$ orthogonal

2 Points

b) decide if **a** and **c** are parallel

Class Notes
pg 743

If $\vec{a}\times\vec{c}=0 \Rightarrow$ parallel

2 Points

c) calculate the area of the triangle determined by **a** and **b**

Class Notes
Quiz #2

$$\text{Area} = \frac{1}{2}\|\vec{a}\times\vec{b}\|$$

1 2

3 Points

d) calculate the volume of the parallelepiped determined by **a**, **b**, and **c**.

Class Notes
pg 743

$$\text{Volume} = |\vec{a}\cdot(\vec{b}\times\vec{c})|$$

3 Points

20 Points

3. Identify the given surface and convert its equation to cylindrical coordinates: $x^2 + y^2 + z^2 = 2x$.

8 Points

$(x^2-2x+1)+y^2+z^2=1$

$(x-1)^2+y^2+z^2=1$ 4

sphere, center $(1,0,0)$

radius $= 1$

$r^2+z^2=2r\cos\theta$

2 2

Sample Test
Quiz #1
Quiz #3
p730 #57

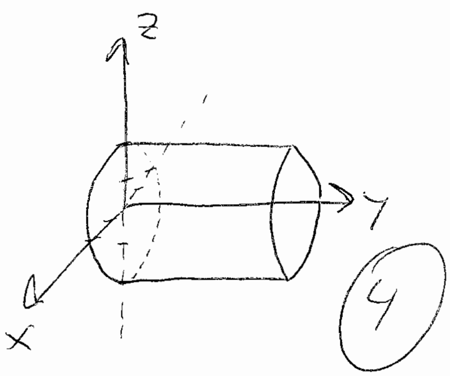
4. Convert the following equation in spherical coordinates to one in rectangular coordinates and sketch the surface: $\rho^2\,[\sin^2\phi\cos^2\theta + \cos^2\phi] = 4$.

8 Points

$x = \rho\sin\phi\cos\theta, \quad z=\rho\cos\phi$

$\Rightarrow x^2+z^2=4$

2 2



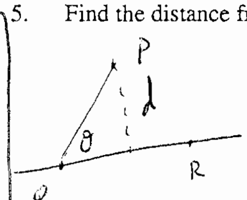
4

Sample Test
Quiz #3
p576
#15, 17

5. Find the distance from the point P(1, 1, 1) to the line through the points Q(0, 6, 8) and R(-1, 4, 7).

12 Points

Class notes
p747
Ex 4
p752
#27-30



$d=|\vec{PQ}||\sin\theta|$ 4

$= \dfrac{|\vec{PQ}\times\vec{QR}|}{|\vec{QR}|}$

$\vec{PQ} = -\hat{i}+5\hat{j}+7\hat{k}$ 1

$\vec{QR} = -\hat{i}-2\hat{j}-\hat{k}$ 1

$\vec{PQ}\times\vec{QR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 5 & 7 \\ -1 & -2 & -1 \end{vmatrix} = \hat{i}(9)-\hat{j}(8)+\hat{k}(7)$ 3

$d=\dfrac{\sqrt{81+64+49}}{\sqrt{1+4+1}} = \sqrt{\dfrac{194}{6}} = \sqrt{\dfrac{97}{3}}$ 3

$=5.68$

line $x=0-t$

$y=6-2t$

$z=8-t$ 6

$d^2=(1--t)^2+(1-6+2t)^2+(1-8+t)^2$

$=1+2t+t^2+25-20t+4t^2+49-14t+t^2$

$=6t^2-32t+75$

$d\,d^2/dt = 12t-32=0$

$\Rightarrow t=32/12=8/3$ 3

$\Rightarrow d^2=6\cdot(8/3)^2-32(8/3)+75$

$=128/3-256/3+225/3$ 3

$=97/3$

$\Rightarrow d=\sqrt{97/3}$

28 Points

HW #2 problem # 2

6. Find an equation of the plane passing through the point (1, 6, -4) and containing the line $x = 1 + 2t$, $y = 2 - 3t$, $z = 3 - t$.

12 Points

Point on plane is $(1, 6, -4)$

direction of line $= 2\hat{i} - 3\hat{j} - \hat{k}$ (2)

point on line is $t=0 \Rightarrow (1, 2, 3)$

forms the vector $0\hat{i} - 4\hat{j} + 7\hat{k}$ (4)

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -4 & 7 \\ 2 & -3 & -1 \end{vmatrix} = \hat{i}(25) - \hat{j}(-14) + \hat{k}(8) \quad (2)$$

eq plane $25(x-1) + 14(y-6) + 8(z+4) = 0 \Rightarrow 25x + 14y + 8z - 77 = 0$ (4)

7. A particle starts at the point (0, 1, 0) with initial velocity $2\mathbf{j}$. Its acceleration is $\mathbf{a}(t) = t\,\mathbf{i} + e^t\,\mathbf{j} + 2\,\mathbf{k}$. Find the particle's location at time $t = 2$.

10 Points

$\vec{r}''(t) = t\hat{i} + e^t\hat{j} + 2\hat{k}$

$\vec{r}'(t) = (\frac{t^2}{2} + c_1)\hat{i} + (e^t + c_2)\hat{j} + (2t + c_3)\hat{k}$

$\vec{r}'(0) = 0\hat{i} + 2\hat{j} + 0\hat{k} = c_1\hat{i} + (1 + c_2)\hat{j} + c_3\hat{k} \Rightarrow c_1 = 0,\ c_2 = 1,\ c_3 = 0$

so $\vec{r}'(t) = \frac{t^2}{2}\hat{i} + (e^t + 1)\hat{j} + 2t\hat{k}$ (4)

PS 782 # 29

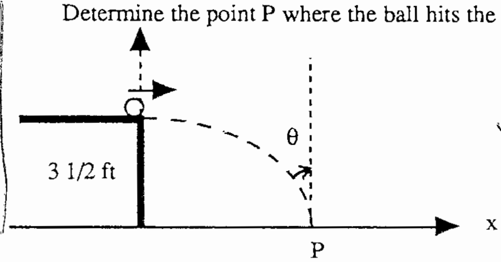
$\vec{r}(t) = (t^3/6 + d_1)\hat{i} + (e^t + t + d_2)\hat{j} + (t^2 + d_3)\hat{k}$

$\vec{r}(0) = 0\hat{i} + 1\hat{j} + 0\hat{k} = d_1\hat{i} + (1 + d_2)\hat{j} + d_3\hat{k} \Rightarrow d_1 = d_2 = d_3 = 0$

So $\vec{r}(t) = \frac{t^3}{6}\hat{i} + (e^t + t)\hat{j} + t^2\hat{k}$ (4), $\vec{r}(2) = \frac{4}{3}\hat{i} + (e^2 + 2)\hat{j} + 4\hat{k}$ (2)

8. As shown in the figure, a ball rolls off of a table with a speed of 2 ft/s. The table is 3 1/2 ft high. Determine the point P where the ball hits the floor. For **six bonus points**, find the angle θ!

Sample Test class notes Quiz #4



$x = v_0 \cos\alpha\, t = 2t$ (2)

$y = -\frac{1}{2}gt^2 + v_0 \sin\alpha\, t + 3\frac{1}{2}$

so $y = -16t^2 + 3\frac{1}{2}$ (2)

$P = 2t$, $0 = -16t^2 + 3\frac{1}{2} \Rightarrow t^2 = \frac{7}{2 \cdot 16} \Rightarrow t = \frac{1}{4}\sqrt{7/2} = 0.4677$ sec (3)

Thus $P = \frac{1}{2}\sqrt{7/2}$ (3)

$$\boxed{\theta = \cos^{-1}\left[\frac{(0\hat{i} + 1\hat{j}) \cdot (2\hat{i} - 32t\hat{j})}{\sqrt{1}\ \sqrt{4 + (32t)^2}}\right] + \pi \ (3) = \cos^{-1}\left[\frac{-8\sqrt{7/2}}{\sqrt{4 + 32 \cdot 7}}\right] + \pi \ (2) = 7.61^\circ \ (1)}$$

or $\tan(\theta + \frac{\pi}{2}) = \frac{dy}{dx} \Rightarrow \theta = \tan^{-1}\frac{dy}{dx} - \pi/2 = \tan^{-1}\frac{dy/dt}{dx/dt} - \pi/2$ (3)

10 Points
6 Bonus

so $\theta = \tan^{-1}\left[\frac{-32t}{2}\right] - \pi/2$ or $\theta = \tan^{-1}(-16t) - \pi/2 = \tan^{-1}\left(\frac{-16}{4}\sqrt{7/2}\right) - \pi/2$ (2)

$\theta = \tan^{-1}(-4\sqrt{7/2}) - \pi/2 = 7.61^\circ$ (1)

38 Points

Sample Test

9. The position of a projectile is given by $\mathbf{r}(t) = 2\mathbf{i} + \ln[\sin(t)]\mathbf{j} + t\,\mathbf{k}$, $\sin(t) > 0$. Find the following :

a) The projectile's velocity, $\mathbf{v}$

$$\vec{r}\,'(t) = \vec{v}(t) = 0\hat{\imath} + \frac{\cos t}{\sin t}\hat{\jmath} + \hat{k}$$

1 1

2 Points

b) The speed of the projectile

$$|\vec{v}(t)| = \frac{ds}{dt} = \sqrt{\cot^2(t) + 1} = \csc(t)$$

1 1

2 Points

c) The unit tangent vector, $\mathbf{T}$, to the curve

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{0\hat{\imath} + \cot(t)\hat{\jmath} + 1\hat{k}}{\csc(t)} = 0\hat{\imath} + \cos t\,\hat{\jmath} + \sin t\,\hat{k}$$

1 1

2 Points

d) The unit normal vector, $\mathbf{N}$, to the curve

$$\vec{T}\,'(t) = 0\hat{\imath} - \sin t\,\hat{\jmath} + \cos t\,\hat{k} \quad 2$$

$$|\vec{T}\,'(t)| = \sqrt{0 + \sin^2 t + \cos^2 t} = 1 \quad 2$$

$$\text{so } \vec{N}(t) = 0\hat{\imath} - \sin t\,\hat{\jmath} + \cos t\,\hat{k}$$

4 Points

e) The curvature, κ, of the curve

$$\kappa = \left|\frac{dT}{ds}\right| = \frac{|\vec{T}\,'(t)|}{|ds/dt|} = \frac{1}{\csc t} = \sin(t)$$

3 1

4 Points

f) The projectile's acceleration written in terms of $\mathbf{T}$ and $\mathbf{N}$. You do not have to write out $\mathbf{T}$ and $\mathbf{N}$ in your answer.

$$\vec{a} = a_T\vec{T} + a_N\vec{N}$$

$$= s''\,\vec{T} + (s')^2\kappa\,\vec{N}$$

1 1

$$s' = \csc t$$

$$s'' = -\csc t \cot t$$

6 Points

$$= -\csc t \cot t\,\vec{T} + \csc^2 t \cdot \sin t\,\vec{N}$$

$$= -\csc t \cot t\,\vec{T} + \csc t\,\vec{N}$$

2 2

20 Points